Seminar on Motives
Standard Conjectures

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This talk will briefly remind you of the Weil conjectures and then proceed to talk about the Standard Conjectures on algebraic cycles and their relations. We will follow André section 5 and Kleiman's article in the Motives volume.

Some additional recommended sources for the same material:

- Secret Blogging Seminar: "The Weil Conjectures ..."
- MathOverflow: "How would a motivic proof of the Riemann Hypothesis...?"

1 Weil Conjectures

In 1949, André Weil proposed the following set of conjectures about counting points of varieties over finite fields its relations to the cohomology of the variety.

Theorem (Deligne '74 (Weil Conjectures)). Let X be a smooth projective variety of dimension n over a finite field $\mathbb{F}_q$ with Zeta function $Z(X,t) := \exp\left(\sum_{r=1}^{\infty} \#X(\mathbb{F}_{q^r})\frac{t^r}{r}\right) \in \mathbb{Q}[[t]]$ and self-intersection of the diagonal $E := (\Delta_X \cdot \Delta_X) \in \mathbb{Z}$. Let furthermore $h^i(X)$ be the Betti numbers and $\chi(X) := \sum(-1)^i h^i(X)$ the Euler characteristics. Then

1. Rationality: $Z(t) \in \mathbb{Q}(t)$.
2. Functional Equation: $Z(q^{-n}t^{-1}) = \pm q^{nE/2} t^E Z(t)$.
3. Riemann Hypothesis: $Z(t) = \prod_{i=0}^{2n} P_i(t)^{(-1)^{i+1}}$ with $P_i(t) = \prod(1 - \alpha_{ij}t) \in \mathbb{Z}[t]$ such that α_{ij} are algebraic with $|\alpha_{i,j}| = q^{i/2}$.
4. Betti Numbers: $E = \chi(X)$ and $\deg P_i = h^i(X)$.

Weil had tested his conjectures on curves (where a proof using the Jacobian is feasible).

Rationality can be proved with elementary methods (as well as with a cohomological machinery), as done by Dwork in 1960. Grothendieck proved the functional equation in 1965, using Poincaré duality. The Riemann Hypothesis turned out to be the most difficult part.

In 1960, Serre wrote a letter about Kähler analogues of the Weil conjectures. Grothendieck saw that one could copy the proof for curves if one would have some analogue of the Hodge Index Theorem in the finite field setting. This led him to the Standard Conjectures on Algebraic Cycles in 1969. Indepentenly, Bombieri has worked out similar conjectures on algebraic cycles.

Deligne later proved the Weil Conjectures (and much more) without using (and without proving) the Standard Conjectures, by proving a weak Lefschetz theorem for étale cohomology. It was already known that the existence of a Weil cohomology theory for varieties over finite fields would suffice, and as a result of Deligne's work, étale cohomology is such a Weil cohomology theory.

2 Standard Conjectures

2.1 Overview

The Standard Conjectures on Algebraic Cycles are formulated for a fixed Weil cohomology theory $H^\bullet : \mathcal{P}(k) \to K - Vect^{\mathbb{Z}_{\geq 0}}$ with coefficients K of characteristics 0. Whenever an operator on cohomology is said to be "algebraic", this means that it is given as an algebraic cycle (a correspondence).

The names and shorthands for the Standard Conjectures:

A "Hard Lefschetz on Cycles"

$$A(X): \qquad \cdot\omega^{n-2k} : CH^r(X) \xrightarrow{\sim} CH^{n-r}(X).$$

B "Lefschetz Type Standard Conjecture"

$$B(X): \qquad *_L : \bigoplus_{i,r} H^i(X)(r) \to \bigoplus_{i,r} H^i(X)(r) \qquad \text{is algebraic.}$$

C "Künneth Type Standard Conjecture"

$$C(X): \qquad \pi_X^i : H^\bullet(X) \twoheadrightarrow H^i(X) \hookrightarrow H^\bullet(X) \qquad \text{is algebraic.}$$

D "Homological and Numerical Equivalence Coincide"

$$D(X): \qquad \sim_{hom,\mathbb{Q}} = \sim_{num,\mathbb{Q}} .$$

I "Hodge Type Standard Conjecture"

$$I(X): \text{ the } \mathbb{Q}\text{-valued quadratic form } \alpha \mapsto \langle \alpha, *_L(\alpha) \rangle \text{ on } Z^\bullet_{hom}(X)_{\mathbb{Q}} \text{ is positive definite.}$$

We will now review the Lefschetz theorems and then discuss the conjectures.

2.2 Reminder on Lefschetz Theorems

Let $X \hookrightarrow \mathbb{C}P^N$ be an n-dimensional complex variety. Let $Y \hookrightarrow X$ be a smooth hyperplane section. The cohomology groups in this subsection are singular cohomology.

Theorem 1 (Lefschetz Hyperplane Theorem (weak Lefschetz)). If $U := X \setminus Y$ is smooth, then the natural map

$$H^k(X;\mathbb{Z}) \to H^k(Y;\mathbb{Z})$$

is an isomorphism for $k < n-1$ and surjective for $k = n-1$.

We can take $\omega := [Y] \in CH^1(X)$ or even $\omega \in H^2(X,\mathbb{R})$ and consider the **Lefschetz Operators**

$$L : CH^r(X) \to CH^{r+1}(X),\ \alpha \mapsto \alpha \cdot \omega$$

$$L : H^k(X;\mathbb{R}) \to H^{k+2}(X;\mathbb{R})(1),\ \alpha \mapsto \alpha \cup \omega$$

Theorem 2 (Lefschetz vache (hard Lefschetz)). If X is nonsingular then

$$L^k : H^{n-k}(X;\mathbb{R}) \xrightarrow{\sim} H^{n+k}(X;\mathbb{R})(k),\ \alpha \mapsto \alpha \cup \omega^k$$

Hard Lefschetz implies Injectivity in Weak Lefschetz, but in general the two theorems don't imply each other (despite the suggestive naming).

Theorem 3 (Hodge Index Theorem). The signature of the bilinear form $Q(\alpha,\beta) := \int_X \alpha \cup \beta$ on $H^n(X,\mathbb{C})$ is $\sum_{a,b}(-1)^a h^{a,b}(X)$.

One can do this not only for a smooth hyperplane section but any ample divisor D on X with cohomology class η, which is then called a **polarization**. Although the Standard Conjectures a priori depend on this chosen polarizations, only the Hodge Type Conjecture really depends on it.

Remember that weak and hard Lefschetz are part of the definition of a Weil cohomology theory.

2.3 Lefschetz Standard Conjecture

On $H^i(X)$, for $0 \leq i \leq n$ define $\Lambda := (L^{n-i+2})^{-1} \circ L \circ L^{n-i}$, this is an operator $H^i(X) \to H^{i-2}(X)(-1)$. Since L^{n-i} and L^{n-i+2} are isomorphisms, Λ is by definition Lefschetz-inverse to L. Analogously, define for $0 \leq i \leq n$ the operator $\Lambda : H^{2n-i+2}(X) \to H^{2n-i}(X)(-1)$ by $\Lambda := (L^{n-i+2})^{-1} \circ L \circ L^{n-i}$.

With $P^i(X) := Ker(L|H^i(X))$ the primitive elements, we have $H^i(X) = P^i(X) \oplus LH^{i-2}(X)(-1)$, hence

$$H^i(X) = \bigoplus_{j=0}^{n} L^j P^{i-2j}(X)(-j)$$

and from this decomposition

$$H^i(X) \ni x = \sum_{j \geq \max(i-n,0)} L^j x_j \qquad x_j \in P^{i-2j}(X)(-j)$$

we have $\Lambda x = \sum_{j \geq \max(i-n,0)} L^{j-1} x_j$.

We can similarly define another operator, the Lefschetz star:

$$*_L x := \sum_{j \geq \max(i-n,0)} (-1)^{(i-2j)(i-2j+1)/2} L^{n-i+j} x_j$$

then (thanks to the sign) we get $*^2 = \mathrm{id}$ and $\Lambda = *L*$.

While we cannot ask for L to be algebraic (try it!) we can ask whether Λ or $*_L$ are algebraic, i.e. we can ask if there is an algebraic cycle $\lambda \in CH^{n-1}(X \times X)$ such that $\gamma(\lambda) \in H^{2n-2}(X \times X)(n-1)$ induces Λ under the isomorphism

$$H^{2n-2}(X \times X)(n-1) \simeq \bigoplus_{p+q=2n} Hom(H^p(X), H^{q-2}(X)(-1)).$$

One can ask for the inverse of the isomorphism L^{n-i} to be algebraic by a correspondence θ^i. This would follow from the algebraicity of Λ by putting $\theta^i := \Lambda^{r-i}$ and it would imply (by induction) that

$$\pi^i = \theta^i \left(1 - \sum_{j>2n-i} \pi^j\right) L^{r-i} \left(1 - \sum_{j<i} \pi^j\right)$$

is also algebraic, which is the Künneth Standard Conjecture.

2.4 Hodge Standard Conjecture

Let $0 \leq i \leq n/2$. We can intersect the primitive part $P^{2i}(X)(i) \subset H^{2i}(X)(i)$ with the image of the cycle class map $CH^i(X) \to H^{2i}(X)(i)$, call the result $C^i(X)$ and for $x, y \in C^i(X)$ we define $\langle x, y\rangle := (-1)^i \left(L^{n-2i} x \cup y\right)$. This is a $\mathbb{Q}$-valued pairing, and the Hodge Standard Conjecture says that this pairing is positive definite (for all $i \leq n/2$).

Conjecture $A(X, L)$ says that $L^{n-2i} : CH^i(X) \to CH^{n-i}(X)$ is an isomorphism. Conjecture $D(X)$ says that every algebraic cycle $Z \in CH^i(X)$ (any i) which is numerically equivalent to 0 has cycle class $\gamma(Z) = 0 \in H^{2i}(X)(i)$. Under the Hodge Standard Conjecture, these two conjectures are equivalent, as we will see.

Assume $A(X, L)$ and the Hodge Standard Conjecture. Then $*_L : CH^{n-i}(X) \to CH^i(X)$, since $*_L$ is just $\pm L^{2i-n}$. The Hodge Standard Conjecture tells us that for $x \in C^{n-i}(X)$, for $i \geq n/2$, we have $(-1)^i \left(L^{n-2i} x \cup x\right) \geq 0$ and equality only if $x = 0$. We define a bilinear form on $CH^{n-i}(X)$ by $(x, y) \mapsto (x \cdot (*_L y))$. For $x \in C^{n-i}(X)$, we defined $*_L x = (-1)^{2i(2i+1)/2} L^{n-2i} x$, so on $C^{n-i}(X)$ we have $*_L = (-1)^i L^{n-2i}$, hence the two bilinear forms agree. We see that $CH^i(X) \times CH^{n-i}(X) \to CH^n(X) \xrightarrow{tr} \mathbb{Q}$ is non-degenerate, since after the isomorphism L^{n-2i} this is a positive definite form. This shows that a cycle that is numerically equivalent to 0 is already 0, hence homologically 0.

In the other direction, assume the Hodge Standard Conjecture and $D(X)$, then the canonical pairing $CH^i(X) \times CH^{n-i}(X) \to CH^n(X) \xrightarrow{tr} \mathbb{Q}$ is non-degenerate. The linear map $L^{n-2i} : CH^i(X) \to CH^{n-i}(X)$ is injective (because of strong Lefschetz) and bijective because dimensions match (via the canonical pairing). Here, we implicitly used finite-dimensionality of $CH^i(X)$,

Assuming the Lefschetz and the Hodge type Standard Conjectures, one can prove that the Betti numbers don't depend on the cohomology theory and that the characteristic polynomial (on cohomology) of any correspondence has integer coefficients.

2.5 Implications between the conjectures

Special cases are already known:

B is known for curves, surfaces and Abelian varieties.

C is known for classical cohomology theories H over finite fields k, and also for Abelian varieties.

I is known for classical cohomology theories H over field k of characteristics 0 (where it is called "Hodge Index Theorem").

The graph of implications between the standard conjectures for a variety X:

$$\begin{tikzcd}
 & B(X^n) & & \\
A(X\times X) \arrow[r] & B(X) \arrow[u] \arrow[rr] \arrow[dl] \arrow[ddr, dash] & & C(X) \\
A(X) & & & \\
 & & \cdot \arrow[dl, dash] \arrow[dr] & D(X\times X) \arrow[uull] \\
 & I(X,\eta) & & D(X)
\end{tikzcd}$$

The graph of implications between the standard conjectures (for all varieties):

$$\begin{tikzcd}
A \arrow[r, leftrightarrow] & B \arrow[rr] \arrow[dr, dash] & & C \\
 & & \cdot \arrow[r] \arrow[dl, dash] & D \arrow[ull] \\
 & I(X,\eta) & &
\end{tikzcd}$$

In characteristics 0, all standard conjectures follow from the Leftschetz type standard conjecture D.

2.6 Standard Proof of the Weil Conjectures

Really, we're just proving some form of the Riemann Hypothesis for Varieties over finite fields, assuming the Hodge and Lefschetz type Standard Conjectures.

As we have seen, it follows from the Standard Conjectures (or just conjecture D together with Jannsen's result about semisimplicity of numerical motives) that homological motives form a semisimple Tannakian category. Under the Standard Conjectures, this is even a polarizable Tannakian category, which means that there exists a nondegenerate sesquilinear form $\varphi(x, y)$ on every motive X (called **Weil form**), in particular an involution (transposition for the form), such that $\forall u \in End(X) \setminus \{0\} : Tr_X(u \circ u^t) > 0$. This is the key to prove the Riemann Hypothesis for motives, since it allows us to copy the proof for elliptic curves.

Proposition 1. *Let $X \in Mot(\mathbb{F}_q)$ be of weight m with Frobenius endomorphism $F \in End(X)$. Then*

- *$F \cdot F^t = q^m$ (hence $\mathbb{Q}[F] \subset End(X)$ is stable under transposition).*
- *$\mathbb{Q}[F] \subset End(X)$ is a product of fields.*
- *For every homomorphism $\rho : \mathbb{Q}[F] \to \mathbb{C}$, we have $\rho(F^t) = \overline{\rho(F)}$ and $|\rho(F)| = q^{m/2}$.*

Proof. The Weil form $\varphi : X \otimes X \to 1(-m)$ is F-invariant, so

$$\varphi(x, F^t F y) = \varphi(Fx, Fy) = F\varphi(x, y) = q^m \varphi(x, y) = \varphi(x, q^m y)$$

and from non-degeneracy we get $F^t F y = q^m y$.

Now let $R \subset End(X)$ be any transposition-stable commutative subalgebra and $r \in R \setminus \{0\}$. Then $s := rr^t \neq 0$, because $Tr(rr^t) > 0$. From $s^t = s$ we do the same to get $s^2 \neq 0$. If we apply this inductively to s^{2n}, we get that $s^{2n} \neq 0$, and this means s is not nilpotent. Consequently, r is also not nilpotent. The only finite-dimensional commutative $\mathbb{Q}$-algebras without nonzero nilpotents are products of fields.

The algebra $\mathbb{R}[F] := \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}[F]$ is again transposition-stable and a product of fields. The transposition permutes the product factors, but this permutation has to be trivial, because of positivity of the Weil form. On each product factor, the only involutions possible are the identity(on the real factors) and complex conjugation (on the complex factors). Thus $\rho(F^t) = \overline{\rho(F)}$. From the first part we get now

$$\rho(FF^t) = \rho(F)\rho(F^t) = \rho(F)\overline{\rho(F)} = |\rho(F)|^2 = q^m.$$ □