Seminar on Motives
Pure Motives

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In this talk, I want to reach three distinct goals. First, I want to explain why we have to make the book-keeping about Tate twists. Then I will describe why one would want idempotent morphisms to split and how to modify a category such that this holds, a process then used to define pure effective motives, which will then yield the category of pure motives by inverting the Tate twist. Finally, I want to compute the motive of projective space. If there is still time left, we will look at Jannsen's proof of semisimplicity of numerical motives.

1 Preliminaries

1.1 Tate twist

We compare the groups $\mathbb{Z}/5\mathbb{Z}$ and $\mu_5 := \{z \in \mathbb{C} \mid z^5 = 1\}$. By the choice of any $\zeta_5 \in \mu_5 \setminus \{1\}$ we can write down an isomorphism

$$\mathbb{Z}/5\mathbb{Z} \to \mu_5,\ \overline{k} \mapsto \zeta_5^k$$

which depends on the choice of ζ_5. On μ_5 we have a canonical action of $G := Gal(\mathbb{Q}[\sqrt[5]{1}]/\mathbb{Q})$ permuting the roots of $z^5 - 1$. Via any isomorphism this gives a G-action on $\mathbb{Z}/5\mathbb{Z}$, but different isomorphisms yield different G-actions. To keep track of the G-action it is common to write $\mathbb{Z}/5\mathbb{Z}(-1) := \mu_5$ and $\mathbb{Z}/5\mathbb{Z}(-r) := (\mu_5)^{\otimes r}$.

Taking the limit we get $\mu_{5^\infty} := \lim_n \mu_{5^n} = \lim_n \mathbb{Z}/5^n\mathbb{Z}(-1)$, and one can compute the ℓ-adic cohomology of projective space, for $\ell = 5$, which is $H^2_5(\mathbb{P}^1) = \mu_{5^\infty} \otimes \mathbb{Q}_\ell$. The ℓ-adic cohomology carries a Galois action, which in this case is the Galois action on μ. One can also compute $H^0_5(\mathbb{P}^1) = \mathbb{Z}_5$, where the Galois action is the trivial action.

Even if we don't care about the Galois action, any isomorphism $H^0(\mathbb{P}^1) \xrightarrow{\sim} H^2(\mathbb{P}^1)$ is non-canonical and involves choosing a basis for each vector space (in this case, an element in each).

This is why we keep track of the Tate twist.

1.2 Splitting idempotents

Suppose you have an abelian category, then every idempotent endomorphism p of some object X (i.e. $p^2 = p$) has a kernel and an image, and $\mathrm{id} - p$ is also an idempotent, whose image is the kernel of p and its kernel is the image of p, so the short exact sequence $pX \to X \to X/pX$ splits with $X/pX \simeq (\mathrm{id} - p)X$, hence $X = pX \oplus (\mathrm{id} - p)X$.

If you just have an arbitrary category, there is no hope that an idempotent p even has a kernel. Given a preadditive category $\mathcal{A}$, i.e. a category enriched over abelian groups, one can construct a category $\mathcal{A}^\natural$ and an embedding $\mathcal{A} \to \mathcal{A}^\natural$ with the universal property that $\mathcal{A}^\natural$ is **pseudo-abelian**, which means every idempotent has a kernel, and every additive functor $F : \mathcal{A} \to \mathcal{B}$ into another pseudo-abelian category $\mathcal{B}$ factors through $\mathcal{A}^\natural$.

Define the objects of $\mathcal{A}^\natural$ to be pairs (X, p) with $X \in \mathcal{A}$ and $p \in End_{\mathcal{A}}(X)$ idempotent, and morphisms $Mor((X,p),(Y,q)) := \{f \in Mor_{\mathcal{A}}(X,Y) \mid f = q \circ f \circ p\} = q \circ Mor_{\mathcal{A}}(X,Y) \circ p \subset Mor_{\mathcal{A}}(X,Y)$. It is easy to show the universal property now. We call $\mathcal{A}^\natural$ the **pseudo-abelian envelope** of $\mathcal{A}$.

2 Pure effective motives and pure motives

We will use $\mathcal{P}(k)$ to denote the category of smooth projective varieties over k a field. Throughout the talk, F will be a commutative ring used as coefficients.

Remember that a Weil cohomology theory was a certain type of contravariant monoidal functors $\mathcal{P}(k) \to K\text{-}\mathsf{Vect}^{\mathbb{Z}_{\geq 0}}$ with nontrivial cycle class maps and Poincaré duality. This is the kind of cohomology theory one can use to prove the Weil conjectures, via the Lefschetz fixed point formula. Pure motives are supposed to be a category over which every Weil cohomology factors universally.

For now, let $\sim$ be any adequate equivalence relation on cycles. Our categories of motives will depend on $\sim$.

Definition 1. We define a category of correspondences $C_\sim\mathcal{P}(k)_F$ with objects from $\mathcal{P}(k)$ and morphisms $C_\sim\mathcal{P}(k)_F(X,Y) := Corr^0_\sim(X,Y) = Z^{d_X}_\sim(X \times Y)$, where the 0 indicates that the correspondence should be of codimension d_X, i.e. dimension of the target Y.

We get a contravariant functor $\mathcal{P}(k) \to C_\sim\mathcal{P}(k)_F$ by mapping a morphism $f : X \to Y$ to the transposed graph $f^* := {}^t\Gamma_f \in Z^{d_X}_\sim(X \times Y)$. Note how the transposing, the codimension of the cycle and the contravariance of the functor fit together.

Remark 1. The category of correspondences is an F-linear category, with direct sums $\oplus$ given by disjoint union of varieties. It is also a symmetric monoidal category, with monoidal product $\otimes$ given by product of varieties. These structures are compatible, so that we have a $\otimes$-category over F. The functor from $\mathcal{P}(k)$ is a monoidal functor.

2.1 Pure effective motives

Remember that in graded vector spaces, all idempotents split. That's already a good reason for wanting this property for our category of pure motives.

Definition 2. The pseudo-abelian envelope $M^{eff}_\sim(k)_F := (C_\sim\mathcal{P}(k)_F)^\natural$ of the degree-0 F-linear correspondence category of $\mathcal{P}(k)$ is called **pure effective $\sim$-motives** over k with F-coefficients.

We denote the composite functor $\mathcal{P}(k) \to C_\sim\mathcal{P}(k)_F \to M^{eff}_\sim(k)_F$ by h and write for any smooth projective variety X and a correspondence idempotent p of X just $ph(X)$ for the object (X,h) in $M^{eff}_\sim(k)_F$. For morphisms of varieties $f : X \to Y$ we write $f^* : h(Y) \to h(Y)$ instead of $h(f)$.

Now we will see what we get from the pseudo-abelian property:

Definition 3. Let $X \in \mathcal{P}(k)$ and $x : \mathrm{Spec}(k) \to X$ a rational point, $c : X \to \mathrm{Spec}(k)$ the structural morphism. Then $p := x \circ c$ is an idempotent endomorphism of X, hence p^* has a kernel in $M^{eff}_\sim(k)_F$ which splits off as a direct summand. It is called the **reduced motive** $\widetilde{h}(X)$, and doesn't depend on the rational point (up to isomorphism). One has a decomposition $h(X) \simeq h(\mathrm{Spec}(k)) \oplus \widetilde{h}(X)$.

Example 1. The projective line has rational points, so we have $h(\mathbb{P}^1) \simeq h(\mathrm{Spec}(k)) \oplus \widetilde{h}(\mathbb{P}^1)$. This should remind you of the definition of the Tate twist for Weil cohomology theories. The motive $\mathbb{L} := \widetilde{h}(\mathbb{P}^1)$ is called the **Lefschetz motive**, by $1 := h(\mathrm{Spec}(k))$ we denote the monoidal unit object.

2.2 Pure motives

Remember that Weil cohomology theories had Poincaré duality, and one had to keep track of the Tate twists. In particular, there was not only $H^2(\mathbb{P}^1) = 1(-1)$ and its tensor powers, but also the dual $1(1)$. So far, we have no duals in the category of pure effective motives, and there is no object like $1(1)$. However, since one has a cancellation property $Mor(X, Y) \simeq Mor(X \otimes \mathbb{L}), Y \otimes \mathbb{L})$ in pure effective motives, one can formally invert the functor of tensoring with the Lefschetz object $\mathbb{L}$ to obtain a category of pure motives, which is rigid in the Tannakian sense. We will proceed more directly, constructing a category by hand and then showing that $\otimes\mathbb{L}$ is an invertible functor.

Definition 4. Let $M_\sim(k)_F$ be the category whose objects are triples (X, p, r) with $X \in \mathcal{P}(k)$, $ph(X) \in M^{eff}_\sim(k)_F$ and $r \in \mathbb{Z}$, and whose morphisms $Mor((X, p, r), (Y, q, s))$ are given by correspondences of degree $s - r$, i.e. $Mor((X, p, r), (Y, q, s)) \subset q \circ Z^{d_X - r + s}_\sim(X \times Y) \circ p$.

We denote $ph(X)(r) := (X, p, r)$. If $r = 0$ we leave out (r), if $p = \mathrm{id}$ we leave out p. We embed $M^{eff}_\sim(k)_F \to M_\sim(k)_F$ fully by this notation. The object $1(1) = (\mathrm{Spec}(k), \mathrm{id}, 1)$ is called **Tate motive**.

Proposition 1. *In $M_\sim(k)_F$ we have a canonical isomorphism $1(-1) \simeq \mathbb{L}$ and $\mathbb{L}$ has no automorphisms.*

Proof. We will use, in the proof, that $Z^1_{\sim rat}(\mathbb{P}^1) \simeq F$, generated by the class of any rational point of $\mathbb{P}^1$, and in turn p^* will act like id on this set. Then we inspect closely:

$$Mor(1(-1), \mathbb{L}) = Mor\left((\mathrm{Spec}(k), \mathrm{id}, -1), (\mathbb{P}^1, \mathrm{id} - p^*, 0)\right)$$

$$= (\mathrm{id} - p^*) \circ Corr^1_\sim(\mathrm{Spec}(k), \mathbb{P}^1)_F \circ \mathrm{id} = (\mathrm{id} - p^*) \circ Z^1_\sim(\mathbb{P}^1)_F = \{*\}.$$

By the same reasoning, $Mor(\mathbb{L}, 1(-1)) = \{*\}$ (a singleton set). Postcomposing these two morphisms $1(-1) \leftrightarrow \mathbb{L}$ with any automorphisms gives nothing new, so there are no automorphisms except the identity. In particular, $1(-1) \leftrightarrow \mathbb{L}$ are inverse to each other. □

Proposition 2. *In $M_\sim(k)_F$ one has duals $h(X)^\vee \simeq h(X)(d_X)$ and $1(-1)^\vee \simeq 1(1)$, hence internal Homs, and the category is rigid.*

This is purely formal.

Remark 2. There are some interesting $\otimes$-subcategories of pure motives. Take the $\otimes$-subcategory $\otimes$-generated by the Tate motive, this is called the category of **pure Tate motives**. The $\otimes$-subcategory $\otimes$-generated by motives of dimension 0 varieties (algebraic field extensions of k) is called the category of **pure Artin motives** $AM(k)_F$. One can show that pure Artin motives don't depend on the equivalence relation $\sim$ any longer and that $AM(k)_F$ is $\otimes$-equivalent to $Rep_F(Gal(\overline{k}/k))$.

3 Universality of Chow motives

For $\sim$ the rational equivalence relation, we call $CHM(k)_F := M_\sim(k)_F$ the category of **Chow motives**.

Denote for a very short moment $\mathcal{T} := CHM(k)_F$ and $H := h$. Then we have the following:

Chow motives are a monoidal functor $H : \mathcal{P}(k)^{op} \to \mathcal{T}$ into a rigid pseudo-abelian $\otimes$-category over F, with a $\otimes$-invertible object $\mathbb{L}$ such that $\mathbb{P}^1 \to \mathrm{Spec}(k) \to \mathbb{P}^1$ induces a decomposition $H(\mathbb{P}^1) = 1 \oplus \mathbb{L}$; we have, for all varieties X of pure dimension d_X, a trace map $Tr_X : H(X) \to 1(-d_X)$, monoidal in X, which identifies $H(X)^\vee$ with $H(X) \otimes \mathbb{L}^{\otimes -d_X}$; we also have cycle class maps $\gamma_X^r : CH^r(X)_F \to Mor(1, H(X) \otimes \mathbb{L}^{\otimes -r}$, which are contravariant in X, monoidal (i.e. $c^n_{X\times Y} = \sum_{r+s=n} c^r_X \otimes c^s_Y$) and normalized in the sense that $tr_X \circ \gamma_X^{d_X}$ is the degree map for X of pure dimension d_X.

Theorem 1. If we take the category of all functors $H : \mathcal{P}(k)^{op} \to \mathcal{T}$ with the same properties, with varying $\mathcal{T}$, then $h : \mathcal{P}(k)^{op} \to CHM(k)_F$ is initial in the sense that all other H factor over h, i.e. there are **realizations** $\omega_H : CHM(k)_F \to \mathcal{T}$ such that $H = \omega_H \circ h$.

One can easily show that it is equivalent to give a Weil cohomology theory or to give a $\otimes$-functor from Chow motives with $H^i(1(-1)) = 0$ for $i \neq 2$.

One gets even more: the category of mixed Hodge structures is a possible $\mathcal{T}$ in the theorem, and one can endow Betti cohomology with more structure, to end up in this category. Immediately we have a mixed Hodge realization of Chow motives.

Remark 3. If we impose the additional condition on $\mathcal{T}$ to be the category of $\mathbb{Z}$-graded objects in a Tannakian category $\mathcal{A}$ over a characteristics 0 field $F := K$, such that there are twists $\otimes K(r)$ and $H(1(1)) = K(1)$ in degree -2, then Voevodsky's conjecture $\sim_{\otimes nil} = \sim_{num}$ (which implies the sign conjecture, so that $NM(k)_K$ is $\mathbb{Z}/2\mathbb{Z}$-graded) implies that the universal such $H : \mathcal{P}(k) \to \mathcal{T}$ would be $NM(k)_K$, but with changed sign rule, i.e. $Gr_i(M) \otimes Gr_j(N) \simeq (-1)^{ij} Gr_j(N) \otimes Gr_i(M)$.

4 Chow groups, Cohomology and Motive of projective space

Let $\sim$ denote any adequate equivalence relation on cycles again.

From the intersection theory black box we get the following:

1. $CH^0(\mathbb{P}^1)_F = F$ (i.e. points are not rationally trivial)
2. $CH^\bullet(\mathbb{P}^n)_F = F[H]/(H^{n+1})$, where H is the class of a hyperplane, in degree 1.
3. In $CH^\bullet(\mathbb{P}^n \times \mathbb{P}^n)_F$ the class of the diagonal of $\mathbb{P}^n$ can be decomposed as $[\Delta] = \sum_{i=0}^n H^i \times H^{n-1}$.
4. As graded F-algebras, $CH^\bullet(X \times \mathbb{P}^n) \simeq CH^\bullet(X)_F[H]/(H^{n+1})$ with H in degree 1.

Remark 4. Let H be any Weil cohomology theory with coefficients K. From axioms, we have $H^\bullet(\mathrm{Spec}(k)) = K$. One can always compute $H^\bullet(\mathbb{P}^1) = H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^{\not{K}})$, since Poincaré duality gives us $\dim_K H^0(\mathbb{P}^1) = \dim_K H^2(\mathbb{P}^{\not{K}})$ and the fact that there is no cohomology above dimension 2, and by the Lefschetz fixed point formula, the Euler characteristics is given by $Tr(\gamma(\Delta) \cup \gamma(\Delta)) = \deg(\Delta \cdot \Delta)$, and a quick computation yields 2 (e.g. look at the two-fold cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ with branch locus Δ), so there can be no degree 1 cohomology.

Furthermore, suppose we know how $\omega_H(\mathbb{L})$ looks like. Then we already know $H^\bullet(\mathbb{P}^n)$ for any n, since $H(\mathbb{P}^1) = K \oplus \omega_H(\mathbb{L})$ and $\mathbb{P}^n = Sym^n(\mathbb{P}^1)$, together with the Künneth isomorphism yields $H^\bullet(\mathbb{P}^n) \simeq H^\bullet(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)^{S_n} \simeq Sym^n H^\bullet(\mathbb{P}^1)$. There, to see $H^\bullet(Sym^n \mathbb{P}^1) = H^\bullet(\mathbb{P}^1)^{S_n}$ one needs (as far as I can see) to calculate the Chow ring of $\mathbb{P}^n$, and use cycle class maps, to lift $CH^\bullet(\mathbb{P}^1) \simeq H^{2\bullet}(\mathbb{P}^1)$ to symmetric products.

Lemma 1 (Manin's identity principle)**.** *The functor that maps a motive M to the functor*

$$\omega_M : \mathcal{P}(k) \ni Y \mapsto M_\sim(h(Y), M(*)) := \bigoplus_{r\in\mathbb{Z}} M_\sim(h(Y), M(r))$$

is fully faithful (like the Yoneda embedding), if maps are mapped accordingly. Here, the grading on the right hand side is taken to be the grading in intersection groups, not a new grading introduced by the index r.

Example 2. We compare the motives $h(\mathbb{P}^n)$ and $\bigoplus_{s=0}^n 1(-s)$ by looking at their corresponding functors ω_M. The only input we need from intersection theory is

$$CH^\bullet(X \times \mathbb{P}^n) \simeq CH^\bullet(X)_F[H]/(H^{n+1}) = \bigoplus_{s=0}^n CH^\bullet(X) \cdot H^s.$$

We compute for any $Y \in \mathcal{P}(k)$:

$$M_\sim(h(Y), h(\mathbb{P}^n)(r)) = Z_\sim^{d_Y+r}(Y \times \mathbb{P}^n) \simeq \bigoplus_{s=0}^n Z_\sim^{d_Y+r-s}(Y),$$

$$M_\sim(h(Y), \left(\bigoplus_{s=0}^n 1(-s)\right)(r)) = \bigoplus_{s=0}^n Z_\sim^{d_Y+r-s}(Y).$$

Using the first line with $Y = \mathbb{P}^n$ and $r = 0$ we can take the identity on $\mathbb{P}^n$ to yield a canonical morphism $h(\mathbb{P}^n) \to \bigoplus_{s=0}^n 1(-s)$, which is an isomorphism, since it induces an isomorphism of corresponding functors.

The same example basically works for projective bundles $P \to X$ of rank n, where $h(P) = \bigoplus_{s=0}^n h(X)(-s)$.

From this we can compute any Weil cohomology theory H on $\mathbb{P}^n$, as soon as we have $H(\mathbb{P}^1)$. As soon as we know the cohomology of the base, we can compute the cohomology of any projective bundle.

5 Semisimplicity of numerical motives

Theorem 2 (Jannsen). Let K be a field, then the category of motives $M_\sim(k)_K$ is semi-simple iff $\sim$ is numerical equivalence. In particular, $NM(k)_K$ is semi-simple.

In the proof, adequate equivalence relations are identified with $\otimes$-ideals in $CHM(k)_K$. The largest $\otimes$-ideal corresponds to numerical equivalence. The existence of a Weil cohomology theory $H : CHM(k)_K \to \mathcal{T}'$ gives a functor on which one applies the following Lemma, which is the categorical heart of the proof.

Lemma 2. *Let $H : \mathcal{T} \to \mathcal{T}'$ be a K-linear $\otimes$-functor between rigid $\otimes$-categories over K, with the extra properties that in $\mathcal{T}$, we have $End_{\mathcal{T}}(1) = K$ and in $\mathcal{T}'$ the* Hom*s are finite dimensional and nilpotent endomorphisms have vanishing trace (which is the case for abelian categories). Let $\mathcal{N}$ be the largest $\otimes$-ideal of $\mathcal{T}$. Then:*

1. *The pseudo-abelian envelope of $\mathcal{T}/\mathcal{N}$ is abelian and semi-simple.*
2. *The only $\otimes$-ideal $\mathcal{I}$ of $\mathcal{T}$ for which $\mathcal{T}/\mathcal{I}$ is semi-simple is $\mathcal{I} = \mathcal{N}$.*