Seminar on Motives
Overview

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1 Motivation

1.1 Motivation from Classical Algebraic Topology

Let Top be the category of compactly generated weak Hausdorff spaces (or just CW complexes), then a cohomology theory on Top that satisfies the Eilenberg-Steenrod axioms is given by morphisms into a spectrum, so the H^i are representable functors in the homotopy category. One can ask for a universal cohomology theory in the sense that all other cohomology theories factor over it, and the functor that maps a space to its stable homotopy type is such a universal invariant for cohomology theories.

If we look at all complex orientable cohomology theories, such as complex cobordism and modules under the spectrum MU, we get a different universal problem, and a different universal theory: complex cobordism.

The search for a universal **Weil cohomology theory** on smooth projective varieties is what leads to pure motives. Extending this somehow to not-necessarily smooth-or-projective varieties or schemes leads to mixed motives.

1.2 Motives as Universal Cohomology Theory

A Weil cohomology theory is a collection of contravariant functors H^i from varieties to vector spaces which satisfies a Künneth formula, a Poincaré duality, has a cycle class map, and more. These requirements allow to use a Weil cohomology theory to say something about enumerative geometry and Zeta functions.

Given such a cohomology theory $H : \text{Varieties}/k \to \text{Vect}/k$, we want to factor it over a universal category of Motives:

$$\text{Varieties}/k \xrightarrow{h} \text{Motives}/k \xrightarrow{\text{real}} \text{Vect}/k$$

Clasically, one took h to be contravariant and the realization functor to be covariant, but the modern convention of Voevodsky has h covariant instead.

If we restrict to smooth projective varieties (as one usually does when speaking of Weil cohomology theories), one speaks of the category of **Pure Motives**, otherwise of **Mixed Motives**.

Since the cohomology theories are $\otimes$-functors and the target category of vector spaces is abelian, one can hope for any such category of motives to be an abelian $\otimes$-category as well, or at least some derived version of that.

Actually, the Weil cohomology theories we know admit more structure than just a vector space. Betti cohomology has a Hodge structure and ℓ-adic cohomology has a Galois action. These more refined target categories are abelian $\otimes$-categories, too.

There are more interesting (co)homology theories than just the Weil cohomologies. For example, (higher) Chow groups are some kind of motivic cohomology theory, quite different in spirit. It might be helpful to read the remark of Saito in footnote 28 on page 66 (14) of Barbieri-Viale's "A Pamphlet on Motivic Cohomology".

2 What do we expect?

A motive should decompose into the building blocks of the cohomology. Since the cohomology of an irreducible variety is not necessarily a simple object, we need more to decompose a motive. Such a decomposition corresponds to having projectors onto the components, which are idempotents. So a crucial idea in the theory of motives is to construct a category which has enough idempotents, and make these idempotents actually splitting (this is called pseudoadditivity).

2.1 Algebraic correspondences

One way to get more morphisms than just morphisms of varieties, is to take correspondences instead. Correspondences relate to morphisms as relations relate to functions. Denote by $A^\bullet(X)$ the algebraic cycles (graded by codimension) modulo rational equivalence and by $H^\bullet(X)$ a cohomology theory in finite dimensional vector spaces that has a morphism $A^\bullet(X) \hookrightarrow H^\bullet(X)$ and satisfies Künneth: $H^\bullet(X \times Y) \simeq H^\bullet(X) \otimes H^\bullet(Y)$. Then we can look at

$$A^\bullet(X\times Y) \hookrightarrow H^\bullet(X\times Y) \xrightarrow{\sim} H^\bullet(X)\otimes H^\bullet(Y) \xrightarrow{\sim} H^\bullet(X)^\vee\otimes H^\bullet(Y) \xrightarrow{\sim} Hom(H^\bullet(X), H^\bullet(Y)).$$

If we look at all such functors $H^\bullet$, we could conjecture that $A^\bullet(X\times Y)$, being in every Hom-space, is a "motivic" part. That's a motivation to look at algebraic cycles on $X \times Y$ as morphisms of motives.

2.2 The motive of projective space

The motive of $\mathbb{P}^1$ splits by using any k-rational point, since this gives us the idempotent $\mathbb{P}^1 \twoheadrightarrow \operatorname{Spec} k \hookrightarrow \mathbb{P}^1$.

$$\begin{aligned} &h(\mathbb{P}^1) = h_0(\mathbb{P}^1) \oplus h_2(\mathbb{P}^1), \\ &h_0(\mathbb{P}^1) = h_0(\operatorname{Spec} k) = h(\operatorname{Spec} k) =: \mathbb{Q}(0), \\ &h_2(\mathbb{P}^1) =: \mathbb{Q}(-1)[2] \text{ is called } \textbf{Lefschetz motive}. \end{aligned}$$

The motive $\mathbb{Q}(-1)$ is called the **Tate Motive**.

2.3 Why should one learn motives?

The standard application of motives that one had in mind historically was this: If you know how the motive decomposes, and you know all cohomology of the simple parts, then you can calculate all cohomological invariants directly. We will see more such decompositions in the talk about examples.

However, the method to decompose a motive almost always gives a method to calculate all cohomological invariants, without looking at motives at all. Motives are still a good thing, for example Deligne's early works are attributed by Grothendieck to stem from a "motivic view" at the problem. So, we may not want to work intensely with motives, but we might want to grasp the philosophy. This is why we have a talk about a motivic proof of the Weil conjectures.

3 Various Approaches to Motives

1. Systems of (mixed) Realizations: Take all Weil cohomology theories you know of, with their enriched target category (Mixed Hodge Structures, etc.) and take a big cartesian product. Now you have an abelian $\otimes$-category and a "motive" functor. Heuristically, the conjecture is that this motive functor is fully faithful, if we take just the Weil cohomology theories we know today.

2. The Grothendieck ring of varieties: $K_0(Var_k)$ is the abelian group generated by isomorphism classes of k-varieties under the relation $[X \setminus Y] = [X] - [Y]$ whenever Y is a closed subscheme of X. From the (reduced) fiber product of varieties one gets a ring structure on $K_0(Var_k)$. All isomorphism invariants that respects the relations factor over this ring. This ring feels already "motivic": $[\operatorname{Spec} k] = 1$ and $[\mathbb{P}^n_k] = 1 + [\mathbb{A}^1] + [\mathbb{A}^1]^2 + \cdots + [\mathbb{A}^1]^n$. If two varieties are equal in the Grothendieck ring, their Zeta functions coincide. There is actually an invariant called the **virtual motive**, which assigns to a smooth projective variety the class of its Chow motive in $K_0(Mot_{rat}(k))$, and this virtual motive factors over the Grothendieck ring of varieties.

3. Grothendieck (pure) motives: Take varieties as objects, but correspondences as morphisms. To get a composition law for correspondences, need to impose an equivalence relation with some suitable properties (ultimately, to use intersection theory). The bad thing: there are multiple adequate choices for such an equivalence relation. This leads to Chow motives, homological motives, numerical motives (and more). Conjecturally, all these coincide and have all properties we want for a category of pure motives. What remains to prove? The standard conjectures on algebraic cycles, for example. We will study Grothendieck motives in the next 5 talks.

4. Nori's cohomological (mixed) motives: This category is unconditionally defined as the (localization at $(\mathbb{G}_\mathrm{m}, \{1\}, 1)$ of the) diagram category attached to relative Betti cohomology and the diagram of pairs (X, Y, i) with $Y \subset X$ closed and $i \in \mathbb{Z}$ that are "good", which means that $H^j(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Q}) = 0$ unless $j = i$. The resulting category is rigid Tannakian, which is a certain good kind of abelian $\otimes$-category. The sad thing is, so far we don't know much about morphisms and extensions in Nori's category.

5. Voevodsky's derived (mixed) motives: This is not a $\otimes$-abelian category, but a triangulated category which behaves as if it were the derived category of the imagined abelian category we're looking for. Its construction is possible in various ways, which are all related to $\mathbb{A}^1$-homotopy theory. One knows already a lot of useful information about this category, for example that it contains numerical pure motives as subcategory. One also knows a lot about morphisms and extensions in Voevodsky's category, and its strong relations to (higher) Chow groups and algebraic K-Theory.

Orthogonally to this list, there are also the notions of **Artin motives**, **n-motives**, **Tate motives** and similar other concepts. The concept of an n-motive is plausible in all versions of motives: A motive is an n-motive if it is contained in the subcategory generated by motives of varieties of dimension n. In some sense, looking at n-motives is like looking at the n-skeleta of CW complexes.

A 0-motive is also called Artin motive, and one can also formalize the idea of Artin motives by looking at compatible systems of Galois representations. Similar hands-on approaches via semi-abelian varieties exist for 1-motives.

A Tate motive (as opposed to **the** Tate motive) is any motive that lies in the category generated by the objects $\mathbb{Q}(n)[m]$, which are twists and shifts of **the** Tate motive. The term "category generated by" may be interpreted differently. If one allows non-algebraically closed base fields, the result is a category of **Artin-Tate motives**.

There exist more approaches to do something "motivic", like so-called Hodge cycles. We won't look at them in this seminar.

4 Tannakian Theory, Motivic Galois Groups, Periods

The category of pure motives is supposed to be, conjecturally, an abelian semi-simple tensor category which is Tannakian, which ultimately means it is equivalent to the category of finite representations of a certain linear pro-algebraic group, called the **motivic Galois group**. This, and Nori's construction, are the reason why we will learn Tannakian theory in the following talk.

In the classical theory of periods, a **period** is defined as the value of an integral of an algebraic differential form over an algebraic cycle. The form and the cycle may be defined over the rationals, but the integral gives a complex number. These periods satisfy interesting relations and form a countable ring inside the real numbers. Some transcendent numbers are known to be periods, such as π, $\zeta(n)$ and multiple Zeta values.

Example:

$$\int_{S^1} \frac{\mathrm{d}\, z}{z} = 2\pi i \qquad \text{is a period of the variety } \mathbb{G}_{\mathrm{m}}\,.$$

One can rephrase the definition of a period in cohomological terms: The algebraic cycles are elements in the Betti cohomology of a variety, the differential forms are elements in the algebraic de Rham cohomology. The integration pairing gives an isomorphism (a natural transformation) between the two cohomology functors, defined over the complex numbers. This yields an isomorphism of fiber functors ω_{Betti} and ω_{dR} from the category of motives to vector spaces. In other words, integration is a $\mathbb{C}$-point of the isomorphism scheme $\mathrm{Iso}^{\otimes}(\omega_{Betti}, \omega_{dR})$.

One can show that this isomorphism scheme is a torsor under the motivic Galois group, i.e. after fixing a point like the integration just mentioned, it is isomorphic to the motivic Galois group. Studying the structure of the motivic Galois group can give insight on periods, for example on the dimension of certain subspaces generated by periods, which is a classical theme in transcendence theory. There are some new results in transcendence theory which have been proved motivically, but not otherwise.