Six functor formalism*

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1 The six functors

First of all, this talk takes place in the world of locally compact Hausdorff spaces and sheaves of abelian groups on them, so no algebraic geometry! You can think of smooth manifolds, maybe with some singularities allowed, and sheaves of vector spaces, to be comfortable.

If you really need to think about algebraic geometry, take X an excellent noetherian finite-dimensional scheme over $\mathbb{C}$ and then X^{an} is the kind of object we care about.

1.1 Definitions and first properties

The six functors are:

$$\mathcal{H}om(-,-),\ (-\otimes -),\ f_*,\ f^*,\ f_!,\ f^!.$$

We will talk about the four functors associated to a morphism f, which are pushforward, pullback, proper pushforward, exceptional pullback. We will define these functors on sheaves of abelian groups now.

Definition 1. Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf over X, then we define a presheaf $f_*\mathcal{F}$ over Y by

$$f_*\mathcal{F}U := \mathcal{F}f^{-1}U.$$

Remark 1. Actually, $f_*\mathcal{F}$ is always a sheaf.

For the constant morphism $c : X \to \mathsf{pt}$, the value of $c_*\mathcal{F}$ at the point is just $\Gamma(\mathcal{F})$, the global sections. The right derived functors of Γ are sheaf cohomology $H^\bullet$, so the right derived functors of f_* are something similar. Very short reminder: To compute $H^\bullet$ of $\mathcal{F}$, you take an injective resolution $\mathcal{F} \hookrightarrow I^\bullet$ and then $H^i(\mathcal{F}) = H^i(\Gamma(I^\bullet))$.

The functor of global sections with compact support Γ_c is defined as

$$\Gamma_c(F, U) := \{s \in F(U) \mid \operatorname{supp}(s) \text{ compact}\} \subset F(U) = \Gamma(F, U).$$

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Its right derived functor is called sheaf cohomology with compact support $H_c^\bullet$. To see that the functor Γ_c is useful, remember Poincaré duality for non-compact manifolds X of dimension n:

$$H_c^k(X;\mathbb{R}) \simeq H_{n-k}(X;\mathbb{R}).$$

Verdier duality is going to be a local version of this, where $\Gamma : f_* \widehat{=} \Gamma_c$:?.

Definition 2. Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf over X, then we define a presheaf $f_!\mathcal{F}$ over Y by

$$f_!\mathcal{F}U := \{s \in \mathcal{F}(f^{-1}U) \mid f_{|\operatorname{supp}(s)}\colon \operatorname{supp}(s) \to U \text{ proper}\}.$$

Remember, "proper map" means separated and universally closed, and since we work with Hausdorff spaces, this reduces to "universally closed", which is equivalent to "closed with compact fibres".

Remark 2. Actually, $f_!\mathcal{F}$ is always a sheaf. It is a subsheaf of $f_*\mathcal{F}$ by definition. For proper maps f, we have $f_! = f_*$ by definition.

For the constant morphism $c : X \to \mathsf{pt}$, the stalk of $c_!\mathcal{F}$ at the point is just $\Gamma_c(\mathcal{F})$, the global sections with compact support.

Definition 3. Denote by f^* a left adjoint to f_* and by $f^!$ a right adjoint to $f_!$, whenever these exist. One writes $f^* \dashv f_*$ and $f_! \dashv f^!$ to remember.

Question 1. How do f^* and $f^!$ look like?

Remark 3. We already know that f^* is always given by

$$f^*\mathcal{G}U = \varinjlim \mathcal{G}V, \text{ where the limit ranges over open subsets } V \supset f(U).$$

The functor $f^!$ doesn't even exist in general (on the level of sheaves), but we will see how it looks like in examples where it does exist. We will also see how to circumvent its non-existence by introducing derived categories.

2 Examples with sheaves

We will now look at the four functors in the particularly important example of a closed subspace and its open complement, which often appears in proofs involving stratifications and will be used in the next talk to define the perverse t-structure.

We'll discuss $\mathbb{C}^\times \subset \mathbb{C}$ and what happens with skyscraper sheaves on 0 and local systems on $\mathbb{C}^\times$, and then look at the general situation.

2.1 The complex plane without origin

Now we look at

$$j : \mathbb{C}^\times \hookrightarrow \mathbb{C} \hookleftarrow \{0\} : i$$

where we can explicitly say what happens.

Let A be an abelian group, considered as sheaf over $\{0\}$. Then i_*A is a skyscraper sheaf over the origin and $j^*i_*A = 0$.

Question 2. Local systems on $\mathbb{C}^\times$ are in bijection to representations of the fundamental group. Can we explain what the six functors do to such a local system in terms of this representation?

Let L be a local system on $\mathbb{C}^\times$. Then $j_!L$ and j_*L are still locally constant over $\mathbb{C}^\times$, but no longer on $\mathbb{C}$. We have $i^*j_!L = 0$ and i^*j_*L has the stalk of L over the origin as value (something which can't be described more nicely, I suppose).

2.2 Open and closed immersions

Let $j : U \hookrightarrow X$ be an open embedding and $i : V \hookrightarrow X$ the closed complement.

First of all we observe $i_! = i_*$ and $j^! = j^*$ (the first is clear, for the second you need to prove the adjunction).

We can also describe $i^!$ explicitly here. For a sheaf $\mathcal{F}$ on X, we define a sheaf $\widetilde{\mathcal{F}}$ on X by

$$\widetilde{\mathcal{F}}U := \{s \in \mathcal{F}U \mid \operatorname{supp}(s) \subset V\}$$

and then $i^!\mathcal{F} = i^*\widetilde{\mathcal{F}}$ is a sheaf on V.

Let $\mathcal{F}'$ be a sheaf on V, then we can discuss $i_*\mathcal{F}' = i_!\mathcal{F}'$, which is a sheaf on X with stalks over points $x \in V$ just the stalk $\mathcal{F}'_x$, and for $x \in U$ plain 0. We have $j^*i_!\mathcal{F}' = 0$.

Let $\mathcal{F}''$ be a sheaf on U, then we can discuss $j_!\mathcal{F}''$ and $j_*\mathcal{F}''$. On the points of X which are already in U, they have both the same stalks as $\mathcal{F}''$. On the points of V, $j_!\mathcal{F}''$ has stalks 0, but $j_*\mathcal{F}''$ introduces new stalks on the boundary of U. That's why $j_!$ is called "extension by zero". We have $i^*j_!\mathcal{F}'' = 0$ and also $i^!j_*\mathcal{F}'' = 0$.

We remember

$$j^*i_! = 0, \qquad i^*j_! = 0, \qquad i^!j_* = 0.$$

The functors j^* and i_* are fully faithful, since $j^*j_* = \mathrm{id}$ and $i^*i_* = \mathrm{id}$ are both isomorphisms. We also know $\mathrm{id} = j^!j_!$ (but $i^!i_! \neq \mathrm{id}_{Sh(V)}$), so $j_!$ is fully faithful.

Any sheaf $\mathcal{F}$ on X with $j^*\mathcal{F} = 0$ must have support in V, so it comes from $i_!$. We have some kind of exact sequence:

$$0 \to Sh(V) \xrightarrow{i_!} Sh(X) \xrightarrow{j^*} Sh(U) \to 0.$$

On the level of a single sheaf $\mathcal{F}$ on X, we have exact sequences

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}$$

$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}$$

These exact sequences will be available with derived categories of sheaves as well.

2.3 Why we need the exceptional inverse image

Verdier duality is a sheaf-theoretical generalization of Poincaré duality, originally introduced as **analog of coherent duality for locally compact spaces**. To state it, one needs the exceptional inverse image. Global Verdier duality is just the statement that $f_!$

has a right adjoint in the bounded derived category of sheaves. There is also local Verdier duality, which we don't care about right now.

For a moment, we will take for granted that the objects of the bounded derived category are bounded complexes of sheaves (of abelian groups). This shall suffice to sketch how Verdier duality generalizes Poincaré duality.

Let M be a smooth n-dimensional manifold and $c : M \to \mathsf{pt}$ the constant map. Let k be a field, considered as k-vector space.

We will now take for granted the

Fact 1. If k is considered as sheaf of k-vector spaces over pt, then the **dualizing sheaf** $\omega_M := c^! k$ on M is just $c^! k = k_M[n]$, the sheaf k_M of locally constant k-valued functions on M, placed in homological degree $-n$, thus considered as a complex of sheaves. In the same spirit, $c^! k[-i] = k_M[n-i]$.

This is how Verdier duality gives back Poincaré duality:

Theorem 1. Let $\mathbb{C}[-i]$ the complex of sheaves with skyscraper sheaf $\mathbb{C}$ over pt placed in degree i. Choose an injective resolution $\mathbb{C}_M \hookrightarrow I^\bullet$.

Verdier duality in this situation gives

$$\mathrm{Hom}_{D(\mathsf{pt})}(c_! \mathbb{C}_M, \mathbb{C}[-i]) \simeq \mathrm{Hom}_{D(X)}(\mathbb{C}_M, c^! \mathbb{C}[-i])$$

$$\mathrm{Hom}_{D(\mathsf{pt})}(c_! \mathbb{C}_M, \mathbb{C}[-i]) \simeq \mathrm{Hom}_{D(X)}(\mathbb{C}_M, \mathbb{C}_M[n-i])$$

We can rewrite the Homomorphisms in the derived category as 0-th cohomology of the inner Hom-complex,

$$H^0(\mathcal{H}om^\bullet_{\mathsf{pt}}(c_! I^\bullet, \mathbb{C})[-i]) \simeq H^0(\mathcal{H}om_X(\mathbb{C}_M, \mathbb{C}_M)[n-i])$$

$$H^i((c_! I^\bullet)^\vee) \simeq H^{n-i}(\mathbb{C}_M)$$

$$H^i_c(M; \mathbb{C})^\vee \simeq H^{n-i}(M; \mathbb{C}).$$

For full Verdier duality (and the existence of $f^!$ in general), one needs derived categories.

The term "Six Functor Formalism" refers to an abstract set of structure and axioms, where the existence of the functors, their adjunctions and (global and local) Verdier duality are part of. A good reference is Deglise-Cisinski: "Beilinson motives and the six functors formalism".

3 The bounded derived category and constructibility

Our goal is now to define the bounded derived category with constructible cohomology.

Classically, one would introduce the structure of triangulated category on the homotopy category of chain complexes, and then localize it at quasi-isomorphisms. However, we postpone the technical work with triangulated categories to another talk and focus on a concrete description instead. The technical work is of course necessary, for example to avoid set-theoretical difficulties, but also to avoid having "too many" morphisms.

3.1 Chain complexes of sheaves

Definition 4. In the category of chain complexes, there is a **shift functor**, $[1] : Ch(X) \to Ch(X)$, which maps a chain complex

$$\cdots \to \mathcal{F}^{n-1} \xrightarrow{d^{n-1}} \mathcal{F}^n \xrightarrow{d^n} \mathcal{F}^{n+1} \to \cdots$$

to the chain complex

$$\cdots \to \mathcal{F}^n \xrightarrow{-d^n} \mathcal{F}^{n+1} \xrightarrow{-d^{n+1}} \mathcal{F}^{n+2} \cdots ,$$

i.e. the degree of each entry is changed by -1 and the differential has a changed sign. One can remember the degree shift by the statement $\mathcal{F}[1]^n = \mathcal{F}^{n+1}$.

Definition 5. In the category of chain complexes, we have a **mapping cone construction**, which assigns to every morphism of chain complexes $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ the chain complex

$$C(f)^\bullet := \mathcal{F}^\bullet[1] \oplus \mathcal{G}^\bullet, \qquad d_{C(f)} = \begin{pmatrix} d_{\mathcal{F}[1]} & 0 \\ f & d_{\mathcal{G}} \end{pmatrix}.$$

Fact 2. To every morphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ one can associate a long sequence

$$(*) \qquad \mathcal{F} \to \mathcal{G} \to C(f) \to \mathcal{F}[1] \to \mathcal{G}[1] \to C(f)[1] \to \cdots$$

which gives, under the 0-th cohomology functor, a long exact cohomology sequence.
Sequences like $(*)$ are what is abstracly called **distinguished triangle**.

For a good treatment of triangulated categories and chain complex categories, consider Gelfand-Manin or Schapira or Weibel or Wikipedia. One alternative to using triangulated categories might be to take a model category structure on the chain complexes and localize it, then it gives a model structure whose homotopy category is the derived category.

3.2 The bounded derived category

Definition 6. The (bounded) derived category of sheaves has as objects just (bounded) chain complexes of sheaves. The morphisms are the morphisms from the homotopy category of chain complexes plus some more morphisms, to allow formally inverting quasi-isomorphisms (i.e. morphisms which induce isomorphisms on cohomology). Explicitly, one can think of a morphism (in the derived category) between complexes $\mathcal{F}$ and $\mathcal{G}$ as a zig-zag of morphisms, or a chain of correspondences $\mathcal{L}_i \xleftarrow{s_i} \mathcal{R}_i \xrightarrow{f_i} \mathcal{L}_{i+1}$ with $i = 0, ..., n-1$, $\mathcal{L}_\prime = \mathcal{F}$, $\mathcal{L}_n = \mathcal{G}$ and all s_i are quasi-isomorphisms.

To get this working in a technical sense, one needs to introduce conditions, like

$$s^{-1} \circ s = \mathrm{id} : \qquad \mathcal{F} \xleftarrow{s} \mathcal{G} \xrightarrow{s} \mathcal{F} \quad = \quad \mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}.$$

Remark 4. Take two sheaves $\mathcal{F}, \mathcal{G}$ and place them into homological degree -5 and $+5$, respectively. Then we're talking about $\mathcal{F}[5]$ and $\mathcal{G}[-5]$. In the category of chain complexes, there are no non-zero morphisms between them, but there is, for example, a morphism $\mathcal{F}[5] \oplus \mathcal{G}[-5] \to \mathcal{G}[-5]$. We can actually imagine a chain of correspondences connecting the two. The most natural occurence of such a situation is when you replace a single sheaf $\mathcal{F}$ with an injective resolution $I^\bullet$ which has components in each positive homological degree.

3.3 Existence of the exceptional inverse image functor

The idea to prove existence of $f^! : D^b(Y) \to D^b(X)$ for any $f : X \to Y$, consists of several steps:

- Any contravariant functor $Sh(X) \to \mathbb{Z}\text{-}\mathsf{Mod}$ is representable iff colimits are mapped to limits.
- Any functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ iff $Hom(F(-), D)$ is representable for any $D \in \mathcal{D}$. This is basically the Yoneda Lemma.
- The functor $f_! : D^b(X) \to D^b(Y)$ is isomorphic to a functor $f_\sharp$ such that $Hom(f_\sharp(-), D)$ maps colimits to limits.

It is important to note that not only $f^!\mathcal{F}$ of a sheaf $\mathcal{F}$ is a genuine complex of sheaves, in general, but also that the adjunction holds only in the derived category with the funny morphisms, not in the homotopy category of chain complexes.

3.4 Constructible Complexes

Definition 7. A **stratification** of a space X is a decomposition $X = \coprod X_n$ into finitely many locally closed subspaces called **strata**. One might also choose the subspaces to be constructible and sometimes the exposition prefers an ascending chain of locally closed subspaces $\emptyset = Y_0 \subset Y_1 \subset \cdots \subset Y_n = X$, where we can put $X_i := Y_i \setminus Y_{i-1}$ and $Y_i := \bigcup_{j=0}^{i} X_j$ to switch between the two descriptions.

Remark 5. Over a point, constructibility just means being a finite-dimensional vector space. The ascending chain definition of a stratification indicates how induction over the dimension can be a key argument when using stratifications.

Definition 8. A sheaf $\mathcal{F}$ over X is called **constructible over a stratification** $X = \coprod X_n$, if $\mathcal{F}|X_n$ is a local system, i.e. a locally constant sheaf of vector spaces with finite-dimensional stalks. A sheaf is just called **constructible** if there exists a stratification such that it is constructible over that stratification.

Remark 6. If you take a local system L on $\mathbb{C}^\times$, the pushforward j_*L along $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ is not a local system any longer (the new stalk at the origin obstructs being locally constant). However, the stratification $\{0\} \subset \mathbb{C}$, in other words, $\mathbb{C} = \{0\} \cup \mathbb{C}^\times$, is enough to see that j_*L is still constructible.

Fact 3. The six functors may not preserve local systems, but they preserve constructibility. That's why we're interested in this notion.

Fact 4. The derived category of constructible sheaves is not useful, since the category of (bounded) chain complexes of constructible sheaves doesn't have enough injectives. This means, we must allow to take resolutions by non-constructible sheaves and use constructibility otherwise.

Definition 9. A complex of sheaves is **constructible** if it has constructible cohomology sheaves. It is **constructible with respect to a fixed stratification**, if the cohomology sheaves are constructible with respect to the fixed stratification. Since cohomology isn't changed under quasi-isomorphisms by definition, this yields a subcategory of the bounded derived category, the **bounded derived category with constructible cohomology (and varying stratification)** $D_c^b(X)$.

We might as well fix a stratification of each space once and for all, but then we have the problem that the six functors don't preserve stratifications, in general. There is a way out:

Definition 10. Let X and Y be two smooth manifolds of dimensions i and j, embedded in the same $\mathbb{R}^n$ as disjoint locally closed sets. Then there are two conditions that X, Y can fulfill:

A) Whenever a sequence of points $x_n \in X$ converges to a point $y \in Y$, and the sequence of tangent spaces $T_{x_n}X$ converges to an i-plane T, then T contains the tangent space T_yY.

B) Whenever two sequences $x_n \in X$ and $y_n \in Y$ both converge to a point $y \in Y$, such that the secant lines L_m between x_m and y_m converge to a line L, and the sequence of tangent spaces $T_{x_n}X$ converges to an i-plane T, then T contains L.

(Actually, B implies A).

Now we call a stratification $M = \coprod M_n$ a Whitney-stratification if any M_k and M_l with $M_k \subset \overline{M_l}$ satisfy the conditions A and B, where we must fix an embedding of M in some $\mathbb{R}^n$.

Fact 5. Whitney: Every analytification of an algebraic variety over $\mathbb{C}$ admits a Whitney stratification.

The six functors preserve the bounded derived category with constructible cohomology with respect to fixed Whitney stratifications.

To end the talk, we want to stress that the 6-functor-formalism carries on to the subcategory of the bounded derived category given by the complexes with constructible cohomology. We have a "Recollement" situation again, where the derived category of a space X is described by the subcategory over an open subset and its closed complement. The four functors give a "splitting" of the larger derived category. This will be useful to define the perverse t-structure.