PROGRAMME
SEMINAR ON HOMOGENEOUS AND SYMMETRIC SPACES

GK1821

Programme

Introduction

Groups arise as symmetries of objects and we study groups by studying their action as symmetries on geometric objects, such as vector spaces, manifolds and more general topological spaces. One particularly nice type of such geometric objects are homogeneous spaces.

For example, the general linear group GL_n and the symmetric group S_n arise naturally in many contexts and can be understood from their actions on many different spaces. Already for GL_n there are several incarnations: as finite group $GL_n(\mathbb{F}_q)$, as Lie group $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$, as algebraic group $R \mapsto GL_n(R)$.

As first approximation, we should think of homogeneous spaces as topological coset spaces G/H where H is a subgroup of G. A symmetric space is then a homogeneous space with the property that H is the fixed set of an involution on G. There are other, better and more precise characterizations that we'll use. For example, a Riemannian manifold which is a symmetric space can also be characterized by a local symmetry condition.

Any space with symmetries, i.e. a G-action, decomposes into its orbits, which are each homogeneous spaces. This in itself is the strongest motivation to study

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homogeneous spaces, and it gives reason to not look at homogeneous spaces in isolation, but also in the broader context of G-spaces. Another good reason is that many homogeneous spaces pop up as parameter/moduli spaces. Last but not least, they make good examples to compute something or test a theory.

We will spend some time at the question "what kind of matrix groups are there" since it is one more-or-less answerable version of "what kind of groups are there", and this comes strictly before asking "what kind of homogeneous spaces are there". In fact, the best answer one can look up in the literature is just for homogeneous spaces under matrix groups. Technically behind this is the study of the adjoint action, Lie algebras and root systems on one hand, and forms of varieties on the other (though that aspect will play a minor role for us). It will also show us a close analogy between differential geometry and algebraic geometry. The local structure of homogeneous spaces is explained by slice theorems, and the global structure is often studied by using compactifications, since compact homogeneous spaces are particularly easy to handle.

There are many advanced topics around homogeneous and symmetric spaces we won't be able to discuss due to simple time reasons, but we will try hard to get a good foundation for many topics relevant to the Graduiertenkolleg. Some first outlooks will be presented at the end of the seminar.

We will quickly recollect some formalities ("Basic Definitions"), so that we can spend the seminar time on geometry. This is expected to be more or less well known, although probably in a different presentation than what you're used to. Some of this material was also covered by recent past seminars. Be warned: the style of the following section should not encourage anyone to give talks like this. The following section has precisely the purpose of getting such abstract nonsense out of our way as soon as possible.

Basic Definitions. We assume the participants to be familiar with abstract group actions, the definition of Lie groups, algebraic groups and Lie algebras and (co)tangent spaces in differential and algebraic geometry.

1. *Homogeneous Objects in a Category.* In the following definition, think of $\mathcal{C}$ being the category of topological spaces, manifolds, complex manifolds, algebraic varieties, rigid analytic varieties, sets, simplicial sets or set-valued sheaves on a topological space.

Definition 1. Let $\mathcal{C}$ be a category with products and final object pt and G a group object in $\mathcal{C}$ with multiplication morphism $\mu\colon G \times G \to G$, inverse $i\colon G \xrightarrow{\sim} G$ and neutral element $e\colon \mathsf{pt} \to G$ (the group axioms can be phrased as commutative diagrams). A **left action** of G on an object X of $\mathcal{C}$ is a morphism $\rho\colon G \times X \to X$ which satisfies $\rho(\mu(g,h),x) = \rho(g,\rho(h,x))$. A **right action** of G on X is a morphism $\rho\colon X \times G \to X$ which satisfies $\rho(x,\mu(g,h)) = \rho(\rho(x,g),h)$. An object X with a G-left action is called a G-object, written $G \circlearrowright X$ and a $\mathcal{C}$-morphism $f\colon Y \to X$ between G-objects $G \circlearrowright X$, $G \circlearrowright Y$ is G-**equivariant** if $f(\rho(g,y)) = \rho(g,f(y))$. The G-objects together with the G-equivariant morphisms form a category $G\mathcal{C}$. Via the trivial G-action $\rho = \mathrm{proj}\colon G \times X \to X$, the category $\mathcal{C}$ is a subcategory of $G\mathcal{C}$.

Note that a left action gives rise to a right action by letting the inverse act (and vice versa). Nevertheless, it is important to keep track of the direction of an action, otherwise one gets very wrong formulae.

Definition 2. For any element $x\colon \mathsf{pt} \to X$ we have the **orbit morphism** $o := \rho(\cdot,x)\colon G = G \times \mathsf{pt} \to X$. The kernel of the orbit morphism (i.e. the equalizer of o with $x \circ \pi\colon G \to X$, for $\pi\colon G \to \mathsf{pt}$) is called the **stabilizer** or **isotropy group** $G_x \to G$ at x. If the orbit morphism is an epimorphism, we call the action **transitive**

and (X,x) a **homogeneouos** G-object with basepoint x. If the stabilizer is just the neutral element, X is called a **principal homogeneous** G-object.

Note that a principal homogeneous G-object is isomorphic to G, but this isomorphism depends on the basepoint (this sentence has to be taken with a grain of salt, which concerns the issue of "forms" below; we will sweep it under the rug for now). In this sense, principal homogeneous G-objects are like groups who forgot their neutral element, which is why we won't write $X = G$. In a principal homogeneous object X, one can form quotients $q\colon X \times X \to G$, written $x \setminus y = g$, uniquely defined by $\rho(q(x,y),x) = y$ (i.e. $g.x = y$). Inverses would be $y \setminus e$, which are not available in X due to lack of neutral element.

Example. Affine n-space $\mathbb{A}^n$ is a $(\mathbb{G}_\mathrm{a})^n$-principal homogeneous variety, where $\mathbb{G}_\mathrm{a}$ is the **additive algebraic group** $R \mapsto \mathbb{G}_\mathrm{a}(R) := (R,+)$.

Definition 3. Let $\mathcal{C}$ be a category with products and a topology given by coverings $\{U_i \to X\}$ for each object X. A morphism $E \to X$ is a **fiber bundle** with fiber F if it is locally on X a product, i.e. there is a covering $\{U_i \to X\}$ such that $E|_{U_i} \to U_i$ is isomorphic to $\mathrm{proj}\colon F \times U_i \to U_i$.

If we fix an action $G \circlearrowright X$ (the trivial one, if nothing else is available), a fiber bundle $E \to X$ with action $G \circlearrowright E$ such that $E \to X$ is G-equivariant, is called a G-**bundle**. If the action on one fiber $E|_x$ (therefore on each one) is transitive, we call $E \to X$ a G-**homogeneous bundle**. If the stabilizer of one fiber $E|_x$ in each connected component (therefore of each fiber everywhere) is trivial, we call $E \to X$ a G-**prinicipal homogeneous bundle**.

Note how bundles over the final object pt of $\mathcal{C}$ give back the notion of (principal) homogeneous objects. For a G-bundle, each fiber $E|_x$ is isomorphic to F and carries a G-action, but the different local trivializations may give F different (isomorphic) G-actions. For a principal homogeneous bundle, the fiber is isomorphic to G, but not necessarily in a unique way.

2. *Quotients.* For a topological group G acting on a topological space X, we can always form the quotient $X \twoheadrightarrow X/G$ which can be defined as the coset space with the final topology for the map $X \twoheadrightarrow X/G$, or intrinsically:

Definition 4. An **orbit object** of $G \circlearrowright X$ in $\mathcal{C}$ is the coequalizer of $\mathrm{proj}\colon G \times X \to X$ with the action $\rho\colon G \times X \to X$ (hence a quotient of X), which we write X/G if it exists.

If an orbit object exists, it is unique up to unique isomorphism, but it doesn't exist in general. Moreover, it is very often non-trivial to show existence in a particular case. The easiest example of these problems can already be observed for topological groups: the quotient of a Hausdorff space needn't be Hausdorff again.

If one cannot get orbit objects in a category, it is often desirable to enlarge the category, for example with a universal cocompletion by passing to set-valued presheaves (the theory of stacks, e.g. as used in moduli space theory, concerns how one can still do geometry with certain objects in this larger category).

3. *Forms.* Given a field extension L/K (such as $\mathbb{C}/\mathbb{R}$ or $\mathbb{Q}(i)/\mathbb{Q}$), one can extend a variety X over K to a variety $X_L := X \times_K L$ over L. For an affine scheme $X = Spec(R)$ with $R = K[x_1,\dots,x_n]/(f_1,\dots,f_k)$ this is just $X_L = Spec(R \otimes_K L)$, where $R \otimes_K L = L[x_1,\dots,x_n]/(f_1,\dots,f_k)$.

Given two different (non-isomorphic) rings $R = K[x_1,\dots,x_n]/(f_1,\dots,f_k)$ and $S = K[x_1,\dots,x_n]/(g_1,\dots,g_k)$, it may happen that $R \otimes_K L$ and $S \otimes_K L$ are isomorphic. The isomorphism is allowed to take coefficients in L, so it should not be surprising that there may be no isomorphism over the smaller field K. The same happens for algebraic varieties.

Two varieties X, Y over a field K which happen to be isomorphic over L, i.e. $X_L \xrightarrow{\sim} Y_L$, are said to be **forms** of each other, and X and Y are said to be K-**forms** of X_L.

It may also happen that an algebraic variety X over a field K has no $\mathbb{R}$-points, while some base extension X_L does have points - so that X is not just the empty set. A nice example is the equation $x^2 = -y^2$, which has no solutions in the real numbers but plenty over the complex numbers. The variety $X := Spec(\mathbb{R}[x, y]/(x^2 + y^2))$ has a form $Y := Spec(\mathbb{R}[x, y]/(x^2 - y^2))$, where the isomorphism over $\mathbb{C}$ is given by $x \mapsto x,\ y \mapsto iy$. The form Y has $\mathbb{R}$-points, namely two lines.

This phenomenon is already visible for real and complex Lie groups, and it frequently complicates and enriches the discussion of algebraic groups over more general fields. One can systematically study the forms of a variety by Galois group actions and their cohomology, actually another appearance of principal bundles. We will try not to discuss forms of homogeneous spaces more than necessary for our purposes.

Talks

The first third is about the differential-geometric picture around (mostly compact) Lie groups and Riemannian symmetric spaces. The example of Grassmannians will give us strong motivation because of their parameter space interpretation. Riemannian symmetric spaces will be very important for the outlook, when we study those Riemannian symmetric spaces which have a compatible complex structure - Hermitian symmetric domains. We end the section on Lie groups with an overview of the adjoint action and a discussion of root systems and Dynkin diagrams, an important organising principle for compact Lie groups, linear algebraic groups, their homogeneous spaces and completions.

The second third is about the algebro-geometric picture around (mostly linear) algebraic groups. We will see which place semisimple linear groups (the nice ones) have in the world of algebraic groups, and how to construct the quotient varieties we want to study. The theory of compactifications will be introduced, which will be used in the last part.

The last part is an outlook. First we discuss higher-dimensional uniformization, where homogeneous spaces show up as universal covers. Then we look at period domains in Hodge theory, which is another parameter-space story. After learning a bit about Hermitian symmetric spaces in general, we come to quotients of them, Shimura varieties, which are useful for arithmetic geometry. Finally, there are the affine Grassmannians, which are infinite dimensional homogeneous spaces.

The general idea is that (almost) each talk should take half the time (likely the first part) to present some implementation details, at best the most instructive case of a proof or example for some phenomenon. The other half should be used to present, in a concise way without complete proofs, an overview of the subtopic. This way we will both have some microscopic as well as macroscopic picture of the world of homogeneous spaces. If talks are prepared in pairs, make sure there is one differential-geometric oriented person and one algebro-geometric oriented person in the team, and prepare both parts of the talk together rather than in isolation (this has been a recipe for very good talks in our seminar in the past).

There is not a single reference which covers the whole seminar, although several books seem to try for this. For background reading on differential geometry, [KN63, volume 2, in particular chapters X and XI] fits our purpose (and has a section on Lie groups). For background on algebraic geometry, [Vak13] is a good contemporary reference, and for more information you can always look at [Aut14]. Linear algebraic groups are presented in many textbooks, one which is quite elementary is [Wat79]. The background in Lie theory can be obtained from many books, e.g. [Hum78]. Algebraic homogeneous spaces are discussed in detail in the monography [Tim11].

Here are some additional sources which could be useful for preparation: [Arv03] for a low-level introduction to symmetric spaces, [Bou68] and [Bou75] as a standard reference, [Bum13] for a fast treatment of Lie groups, [Kna02] for a very detailed treatment of Lie groups, [Pro07] also has a nice approach to Lie groups, which seems to be useful for the seminar, [Jos11, Chapter 6] covers the analysis we might need, [Sch89] compares differential and algebraic geometry of homogeneous spaces.

Part I: Lie Groups

1. **Lie Groups.** Give some examples of Lie groups, like S^0, S^1, S^3, $\mathbb{C}^\times$, G_2, Spin groups and whatever comes to your mind.

Mention Hilbert's 5th problem, which asks if group objects in the category of topological manifolds are more than just Lie groups. The answer is no: there exists always exactly one analytic structure on such a group manifold which turns it into a Lie group. Don't attempt to prove this.

Introduce the notions of abelian, simple and semisimple Lie algebras and abelian, semisimple and compact Lie groups, with examples. Give a brief overview on the structure theorems regarding connectedness, simply connectedness, compactness and commutativity.

Briefly sketch how the universal cover of a Lie group gets a Lie group structure and remind us of the exponential map $exp\colon \mathfrak{g} \to G$.

State and explain the Peter-Weyl theorem: Let G be a compact topological group. A matrix coefficient is a function $\varphi : G \to \mathbb{C}$ which factors into a G-representation $G \to GL(V)$ composed with a linear functional $End(V) \to \mathbb{C}$. The set of all matrix coefficients of G is a dense subset of the space of continuous complex functions $\mathcal{C}(G, \mathbb{C})_\infty$ equipped with the supremum norm. It follows from this theorem that any compact Lie group can be embedded in some $U(n)$, so is a matrix group, see [Kna01, Theorem 1.15] and even a Zariski closed group, therefore a linear algebraic group!

Decide for yourself how much of the background for the Peter-Weyl theorem you want to cover (Haar measures and orthonormal bases for $L^2(G)$ will likely be too much to explain in detail).

2. **Quotients.** Discuss the problem of taking the quotient after Lie group actions on manifolds. When/how does one get a (smooth) manifold structure on the quotient space?

You should give some examples that show problems with quotients, like the plane with coordinates x, y and the obvious $\mathbb{Z}/2$-action on the x-coordinate.

Sketch a proof of the slice theorem (or, if you like, special cases of it): Given a manifold M with smooth action by a Lie group G and any point $x \in M$, the orbit map $G \to M,\ x \mapsto gx$ factors through an injective map $G/H \to M$. The theorem states that this map extends to an invariant neighbourhood N of G/H (considered as zero section) in $G \times_H T_xM/T_x(Gx)$ so that it defines an equivariant diffeomorphism $N \xrightarrow{\sim} N' \subset M$ with $Gx \subset N'$.

You could also come up with an example which fails to have slices (you can find such examples on MathOverflow).

As application, show that if a compact Lie group G acts freely on a manifold M, then M/G has a manifold structure. The most important application is the action of a subgroup H of G on G.

You can give some idea what to do if taking the quotient fails. One possibility is the discussion of orbifolds.

The paper [Pal61] on the slice theorem for non-compact Lie group actions is still readable, but there are better references, such as [Mos57] for the first proof for compact Lie groups - and, more reader-friendly, [Lee13].

3. **Grassmannians.** Grassmannians of all sorts play an important role in geometry as parameter spaces. For example, the parameter space for linear subspaces of an affine space $\mathbb{A}^n$ is the projective space $\mathbb{P}^{n-1}$, which is a homogeneous space for GL_n. Another prominent Grassmannian is the flag manifold GL_n/B (where B are the upper triangular matrices in GL_n), which parametrizes complete flags of vector subspaces in $\mathbb{A}^n$. These spaces are both complete, but there are also such parameter

spaces which are not complete: the symplectic Grassmannians, which parametrizes symplectic subspaces, or the oriented Grassmannians are such examples.

Start by discussing abstract parameter problems of flags of subspaces in vector spaces with extra structure (such as a symplectic form, or orientation) with lots of (familiar and less familiar) examples. Explain the group actions on these sets and use the orbit map to get the structure of a homogeneous space on them. Relate the CW complex structure on the homogeneous spaces with the parameter problems. This can also be related to the representation-theoretic interpretation of the cell structures.

Discuss bundles and equivariant bundles over these spaces and (related to that) their interpretation as parameter spaces (classifying spaces) for certain bundles. This also gives us a relation between BG and some G/H.

Discuss the fiber sequences $H \to G \to G/H$ and $G/H \to BH \to BG$. In [MT91, Chapter III], the Serre spectral sequence is used to do some computations cohomology of classical groups and their homogeneous spaces. You can try to explain some of this (without spending too much time on spectral sequences).

You can also mention [MN02], which shows that the spectral sequence methods work even for "homogeneous spaces under loop spaces".

4. **Riemannian Symmetric Spaces.** For this talk, a good discussion of the various aspects (and further references) are to be found in [Esc12], which can be quoted: "Riemannian symmetric spaces are the most beautiful and most important Riemannian manifolds". A detailed monography is [Hel01].

Define symmetric spaces as smooth manifolds which are homogeneous spaces of the form G/H such that there exists an involution on G which fixed points H. Define Riemannian symmetric spaces as symmetric spaces with Riemannian metric or alternatively, as Riemannian manifolds which has at each point an involutive isometry which locally fixes exactly the point.

Give some examples of Riemannian symmetric spaces, such as $\mathbb{R}^n$, S^n, H^n (real hyperbolic space), any compact Lie group (such as $SO(n)$), $SU(n)/SO(n)$, $SL_n(\mathbb{R})/SO(n)$, $\mathbb{O}P^2$ (the Cayley plane) and/or whatever you think could be instructive. Introduce the rank of a symmetric space.

Prove that the two definitions of Riemannian symmetric spaces are equivalent.

While there is still no classification of all symmetric spaces available, one can classify simply connected irreducible Riemannian symmetric spaces. Discuss briefly what kind of classification one gets, in particular the distinction of compact versus noncompact versus euclidean type. The irreducible simply connected symmetric spaces are the real line, and exactly two symmetric spaces corresponding to each non-compact simple Lie group G, one compact and one non-compact. The non-compact one is a cover of the quotient of G by a maximal compact subgroup H, and the compact one is a cover of the quotient of the compact form of G by the same subgroup H. This should also provide motivation for the following talk.

Recall the definition of the curvature tensor of a Riemannian manifold, with some instructive example like the 2-dimensional sphere of radius r. Mention that sectional curvature determines Riemannian curvature and how sectional curvature can be understood intuitively from understanding curved surfaces - to give at least a vague idea what information is captured by the curvature tensor to anyone who hasn't seen it before.

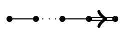
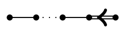
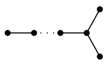
Show that the curvature tensor of a Riemannian symmetric space has to be parallel. Show that a parallel curvature tensor implies that the universal cover is a symmetric space.

After this talk, we will meet Riemannian symmetric spaces again in Part III.

5. **The Adjoint Action.** To understand the vast amount of homogeneous spaces one first has to get an overview of the many groups that may act on spaces. There is a very good classification theory for compact Lie groups and for split semisimple linear algebraic groups, both in terms of the adjoint action, semisimple Lie algebras, root systems and Dynkin diagrams. In some cases, a homogeneous space G/H may then be characterized in terms of the root systems of G and H or even in terms of an annotated Dynkin diagram.

In this talk, an overview of root systems, Dynkin diagrams and the classification of semisimple lie algebras without proof and the consequences for (compact) Lie groups should be presented, starting from the adjoint action.

Root systems in infinite series (all except the exceptional G_2, F_4, E_6, E_7, E_8):

Φ	Dynkin diag.	Lie algebra	Reductive group	Lie group
A_n		$\mathfrak{sl}_{n+1}$	SL_{n+1}	$PSU(n+1)$
B_n		$\mathfrak{so}_{2n+1}$	$SO(n, n+1)$	$SO(2n+1, \mathbb{R})$
C_n		$\mathfrak{sp}_{2n}$	Sp_{2n}	$Sp(2n)$
D_n		$\mathfrak{so}_{2n}$	$SO(n, n)$	$PSO(2n, \mathbb{R})$

(TikZ graphics adapted from Benjamin McKay) About this table: Φ refers to the root system (types are A, B, C, D, E, F, G, the index refers to the rank, which is also the number of big dots in the Dynkin diagram), the Lie algebra is a simple, semisimple one with root system Φ, the reductive group and the Lie group are just some examples that one can keep in mind.

Mention as motivation that the Dynkin diagrams ADE are also known for classifying other mathematical structures, of which you may give some very brief examples.

This talk should be given by someone who knows this theory in and out and can present it in a memorable, pleasant way that helps.

The introduction in [BBCM02, Section III, Chapter 1] gives some overview of topics that would fit in this talk as well.

Part II: Algebraic Groups

6. **Algebraic Groups.** To avoid unnecessary complications, we will only work over perfect fields in this part of the seminar.

Algebraic groups come in two extreme flavours, and the mixture of both: affine and projective. This is made precise by Chevalley's theorem.

Give the formal definition of an algebraic group scheme, a linear (=affine) group scheme and an Abelian scheme. Don't spend too much time on drawing commutative diagrams for the group axioms or the corresponding axioms for Hopf algebras (just mention them). Give some examples, including the additive group $\mathbb{G}_{\mathrm{a}}$, elliptic curves, the multiplicative group $\mathbb{G}_{\mathrm{m}}$, algebraic tori $(\mathbb{G}_{\mathrm{m}})^n$, the general and special linear groups GL_n and SL_n.

Make very clear the difference between an "algebraic torus" $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times = (\mathbb{G}_{\mathrm{m}})^n(\mathbb{C})$ and a "topological torus" $S^1 \times \cdots \times S^1 = T^n$ and a "complex torus" $\mathbb{C}^n/\Lambda$.

Explain the statement of Chevalley's theorem [Con02, Theorem 1.1.] (another proof in [BSU13, Section 2]), but don't attempt to prove it. Don't spend too much time on details of the fppf topology and keep in mind that we will discuss the Nagata compactification later on, so that can be omitted as well.

As an example of "general" algebraic groups appearing in the wild, you can discuss automorphism groups of algebraic varieties.

Give some examples of linear algebraic groups, preferably defined over various base rings. Explain extension of scalars (base change). Introduce restriction of scalars (Weil restriction) and mention (without proof) the important application involving $\mathbb{S} := Res_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_{\mathrm{m}}$, whose real points have a Lie group structure isomorphic to $\mathbb{C}^\times$: the category of real Hodge structures is equivalent to the category of $\mathbb{S}$-representations.

Discuss complete homogeneous varieties under an arbitrary algebraic group: they are always the product of an Abelian variety with a complete homogeneous variety under an affine algebraic group (a flag variety, which is some sort of Grassmannian). This is [BSU13, Theorem 1.3.1 and Theorem 4.1.1] As we will spend some talks on linear groups, their quotients and completions, this should also motivate what's coming next.

7. **Reductive Linear Algebraic Groups.** Define radical and unipotent radical of a linear algebraic group and provide examples.

Define reductive and semisimple linear groups and explain the short exact sequences involving the (unipotent) radical of a group and the quotient, which basically justifies looking at semisimple linear groups and unipotent groups in isolation for many questions.

A short verbal remark on the complications in positive characteristics (reductive vs. linearly reductive) should suffice for our purposes.

Define Borel subgroups and (maximal) tori and give some examples (at best not only SL_n and GL_n; maybe orthogonal groups are nice). Give examples of parabolic subgroups in GL_n.

Explain the statement of the Bruhat decomposition $G = \coprod_{w \in W} BwB$, with a short proof sketch assuming most of the machinery of reductive groups. Connect this to the discussion of CW structures of Lie groups if you know how to.

Sources are [Spr09] or [Bor91] (and many more like these). It would be a good idea to look into [Bri10] to see what the following talk might use.

8. **Quotients by Algebraic Groups.** Explain how Hilbert's 14th problem arose from discussing quotients.

Explain the notion of categorical, geometric, good quotients after an algebraic group action, as defined in [MFK94].

While one could describe a general procedure of taking quotients which leads to stacks that in some rare cases turn out to be algebraic varieties, we will instead first look at some nice cases where we end up with an algebraic variety, with a much easier construction.

Let G be a linear reductive group. For any affine G-variety X, one has a good quotient X/G. In particular, G/H exists for any linear group G with subgroup H. A good presentation of this is [Bri10, Section 1.2].

For H any closed subgroup of a linear group G, the quotient G/H exists and is quasi-projective. This is proved in [Spr09, Chapter 5.5, p.91–97], along other interesting facts about these quotients, for example that the quotient by a normal subgroup is again a linear group.

Call a subgroup $P \subset G$ parabolic if it contains a Zariski-closed maximal connected solvable subgroup (i.e. a Borel). The quotient G/H is a projective variety iff H is a parabolic subgroup (which is why this is often taken as definition; you may also take this as definition, but make sure to state the other characterisation as well). The proof is not difficult, but also not that important for us.

A very comprehensive source on algebraic quotients, from a GIT viewpoint, is [BBCM02, Section I].

If there is still time left, one could explain more of the GIT approach, in particular its motivation and where it is employed.

9. **Compactifications.** The theory of equivariant embeddings of algebraic tori $(\mathbb{G}_{\mathrm{m}})^n$ is the theory of toric varieties, which may also be understood combinatorically by glueing products of affine spaces and tori together. This can be done for a larger class of linear groups than just for tori, but the combinatorics get much more complex. This culminates in the Luna-Vust theory of spherical varieties. A crucial step in the Luna-Vust classification theory is the classification of particularly good embeddings (namely, smooth, complete, and with a good boundary behaviour) of certain well-behaved homogeneous spaces, so-called wonderful varieties. The historically first construction of such good completions were for symmetric spaces, given by De Concini and Procesi. We will not look into Luna-Vust theory or toric varieties in this seminar, but we will briefly discuss wonderful completions in this talk.

Nagata's theorem (not the strongest version) says: for any variety, there is an embedding into a complete variety. There is also a version for morphisms. Sumihiro developed an equivariant version for varieties with an action of a linear algebraic group. In the case that the group is just a torus, one gets more information. Completions are by no means unique, and some are better than others. Some nice classes of equivariant completions are toroidal completions, simple completions and wonderful completions. Toroidal varieties are a slight generalisation of toric varieties, containing flag varieties as well.

For compactification theory, the notions of ample divisors and ample line bundles are important, and in the business of equivariant completions, ample equivariant divisors and G-bundles are as important, so it might make sense to formally define these notions and give some easy examples. Maybe one should even recall the notion of a proper morphism and a complete variety, just to be safe.

The original proofs for Nagata compactification are written with Weil's foundations rather than Grothendieck's foundations for algebraic geometry, so instead one could read [Lüt93] where a modern proof is written up, or the notes of Brian Conrad (unpublished, online) or Deligne's notes as written up by Conrad. For the talk, the proof will be too much, but some ideas and examples might be interesting (that's up to the speaker). Most importantly, the statement of Nagata compactification should be made very clear and the rough idea how ample divisors give projective embeddings.

The original paper [Sum74] on equivariant completions is quite readable, a proof sketch could be developed from it. In particular the local linearizability and the equivariant Chow lemma in that paper might be interesting elsewhere. The rough idea of making a divisor (or a bundle) equivariant and then obtain an equivariant projective embedding should come across.

If you want to talk about toroidal varieties (or toric varieties) you may look at [Tim11, p.174–179], but that is very technical. It might make more sense to just give some examples of toroidal completions, maybe just completions by flag varieties. One thing which we should remember about toroidal completions is that they have a universal property: each equivariant completion is dominated by a toroidal one.

Simple completions are defined by the property that they have only one closed orbit. This nice property can not always obtained, i.e. not every space admits a simple completion. If simple completions exist, then the class of simple and the class of toroidal completions intersect in a single element, which is called the standard embedding. If the standard embedding is smooth, it is called wonderful, and the ambient space in which one embeds is called a wonderful variety. A good source is [Tim11, p.179–188]. We should at least see the statement of [Tim11, Theorem 30.15, p.187], which we can take as intrinsic definition of wonderful varieties. We might also get some idea how Dynkin diagrams can be upgraded to spherical diagrams, which encode spherical systems, which in turn classify wonderful varieties, as stated in [Tim11, Theorem 30.22, p.195]. Please don't attempt to define spherical systems in this talk. Try to avoid defining and discussing the notion of colors as well.

Part III: Special topics

10. **Uniformization.** In differential geometry numerous cases of homogeneous spaces appear as universal covers. For example one can construct models of reals manifolds whose universal cover exhaust all simply-connected projectively-flat bihomogeneous spaces. Restricting ourselves to Kähler geometry, it again transpires that important classes of compact Kähler manifolds (X, w) are uniformized by symmetric spaces. In the absence of the Hermite-Einstein metric on the holomorphic tangent bundle $\mathcal{T}_X$, which is readily available in dimension 1, in higher dimensions one begins the search for an analogous statement (in the sense of uniformization) by asking the Kähler metric to verify the **Einstein condition**. Here, a classical result of Yau (and Miyaoka, in the case of projective varieties) asserts that a natural linear combination of $\mathrm{c}_1^2(X)$ (the square of the first Chern class of $\mathcal{T}_X$) and $\mathrm{c}_2(X)$ (the second Chern class of $\mathcal{T}_X$) is semi-positive with respect to w in the sense of the inequality

$$\int_X \big(2(n+1)\cdot \mathrm{c}_2(X) - n\cdot \mathrm{c}_1^2(X)\big) \wedge [w]^{n-2} \geq 0,$$

which is nowadays referred to as the **Miyaoka-Yau inequality**. Now, it turns out that any compact Kähler manifold X verifying the Miyaoka-Yau **equality** is uniformized by (basic examples of) symmetric spaces, namely the complex Euclidean space (for example when X is Calabi-Yau with vanishing c_2), the projective space and the ball.

11. **Dirac Operators on Homogeneous Spaces.** In this talk, homogeneous differential operators, in particular Dirac operators on homogeneous spaces will be studied. We will hear about the Borel-Weil-Bott theorem.

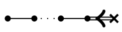
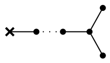
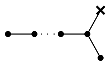
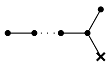
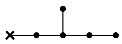
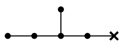
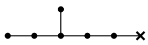
12. **Period Domains.** Period domains are parameter spaces for a polarized Hodge structure of a fixed weight. They are quotients of Lie groups by compact subgroups (but rarely maximal compact subgroups).

In this talk some motivation for looking at period domains should be given, preferably with an easy example. It might make sense to recollect some material on Hodge structures as well.

In [CMSP03], mostly in the third part, there is a lot of material and more references, in particular propositions 4.3.2 and 4.3.3 as well as chapters 4.4, 4.5 and 5.3. Maybe [GS75] will also help.

13. **Hermitian Symmetric Spaces.** Hermitian symmetric spaces are the natural generalization of Riemannian symmetric spaces to complex manifolds.

Briefly discuss the classification of irreducible compact Hermitian symmetric spaces by marked Dynkin diagrams:

A_n		Grassmannian of k-planes in $\mathbb{C}^{n+1}$
B_n		$(2n-1)$-dimensional hyperquadric
C_n		space of Lagrangian n-planes in $\mathbb{C}^{2n}$
D_n		$(2n-1)$-dimensional hyperquadric
D_n		one comp. of the variety of max. dim. null subspaces of $\mathbb{C}^{2n}$
D_n		the other component
E_6		complexified octave projective plane
E_6		its dual plane
E_7		the space of null octave 3-planes in octave 6-space

(TikZ graphics by Benjamin McKay)

Explain as much about noncompact Hermitian symmetric spaces as you like, maybe include Borel embedding, Cartan decomposition, culminating in some overview of the classification of Cartan domains.

Either in this talk or in the following the Baily-Borel or the Borel-Serre compactification and its properties should be mentioned.

14. **Shimura Varieties.** Shimura varieties are quotients of Hermitian symmetric spaces by a congruence subgroup of a reductive algebraic group over a number field, so they are biquotients $\Gamma\backslash G/H$.

First remind the audience briefly of classical modular curves for congruence subgroups of SL_2, to give some motivation and context. Explain the interpretation as moduli space for Hodge structures and the relation to period domains.

Give the definition of a Shimura datum and the associated Shimura variety. Give at least one example.

15. **Affine Grassmannians.** "The" affine Grassmannian is an infinite-dimensional object (an ind-scheme), which can be seen as the flag variety for the group $G(k((t)))$, where G is a reductive linear algebraic group over a field k (so there are really many affine Grassmannians) and $k((t))$ the field of formal Laurent series over k. The group $G(k((t)))$ can be interpreted as a group of loops of $G(k)$.

This talk should introduce affine Grassmannians and discuss their significance in representation theory.

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