Seminar on Motives

Thursdays 14-16 in SR 414
Uni Freiburg

Winter 2012/13*

The theory of **motives**, as envisioned by Grothendieck, stems from various different sources, enumerative geometry, intersection theory, counting solutions of polynomial equations modulo primes, cohomology of algebraic varieties, Galois representations, Hodge structures and several more.

There is a slogan that captures part of the intuition of motives: "Motives are the universal cohomology theory of algebraic varieties" but there is more to it. One can use motivic ideas without using motives at all, the most important idea being that irreducible varieties are no longer atoms, but molecules, which can be (cohomologically) split into parts, which are easier to understand. This intuition is already visible for L-functions, where a lot of the motivation for motives came from.

The current status of the subject is: There is a good notion of **pure motives**, which correspond to smooth projective varieties, although there are some questions left. The abelian category of **mixed motives**, which correspond to all schemes, is yet to be found, although there is a candidate (Nori's cohomological motives). There is a good candidate for the **derived category of mixed motives**, defined by Voevodsky (and equivalently by Levine and Hanamura). This triangulated category is already good for much that one wants to do with the category of mixed motives and it provides the "right" **motivic cohomology**. This setup allowed Voevodsky and others the proofs of the famous Milnor-, Bloch-Kato- and Quillen-Lichtenbaum conjectures (the first resulting in a Fields medal, the other two being even more general).

In this seminar we want to introduce the motivation, basic ideas and formal definitions of pure motives, using some of the prerequisites as black boxes (most notable the inner workings of intersection theory). The goal is threefold: We want to learn tools that are useful elsewhere in algebraic geometry, we want to learn the "motivic" intuition and we want to be able to read the expository literature on the subject afterwards. We will also try to get an idea of mixed motives, using as specific example Hodge theory, which some participants might be more familiar with.

Our guiding examples will be elliptic curves, smooth projective curves and arbitrary curves on one side and projective space and Grassmannians on the other side.

We will require at least some knowledge about varieties, as in Hartshorne's book Chapters I and II, in parts some basic algebraic group theory, and also some basic homological algebra. Each particular talk also has some specific prerequisites.

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Contents

1 Pure Motives

The section on pure motives will mostly follow [And04, Chapters 1–6].

0 Overview (Konrad Voelkel)

In the first talk, an overview on the history and motivation will be given. The relations between the following talks will be explained. There are various articles available that give an overview of the theory. One of the more recent ones is Barbieri-Viale's article [BV05]. Even more recent, but more difficult, are Sujatha's lecture notes [Suj08]. They can also serve as a guideline how to glue the different topics of this seminar together. To a number theorist, [Kim10] might be enjoyable, while a physicist will prefer [RM09]. An overview in french was given by Kahn [Kah06].

1 Tensor and Tannakian Cats (Oliver Straser)

To any finite group (or compact group, or algebraic group, or Lie algebra) one can associate its category of (finite-dimensional) representations. These are tensor-categories, which means they are k-linear monoidal categories with some compatibilities (duality). They also come with forgetful functors to the category of vector spaces. This situation has been axiomatized as Tannakian categories. The most important theorem in this subject is the Tannakian reconstruction theorem: If you take the representation category of an affine pro-algebraic group, the group can be reconstructed from the category as the automorphisms of the forgetful functor. One can also feed all categories that satisfy some Tannakian axioms into the machine, to obtain affine algebraic groups. These are then called fundamental groups or Galois groups, depending on the context. One can understand Grothendieck-Galois theory as example of this idea.

In this talk, the axioms of a neutral Tannakian category have to be explained [DM81, p. 4–29], but there is too much material to present. Most important than a detailed proof of the reconstruction theorem are concrete examples, such as: Take a non-Zariski closed subgroup of GL_n and look at its representation category. Show that it's neutral Tannakian and identify the corresponding affine group scheme as something familiar (Hint: this is what happens in the definition of the Mumford-Tate group from Hodge theory).

It is possible to skip this talk, if one doesn't want to learn anything about motivic Galois groups, and if one is willing to learn the formalism of tensor categories (without Tannakian theory) outside the seminar.

It is conjectured that the abelian category of mixed motives is Tannakian. For certain subcategories of "nice" motives, one can actually show this and study the **motivic Galois groups**, which has already led to some new results in number theory. It is also of great interest to complex geometers, whether there is more structure on the cohomology of a complex variety than the mixed Hodge structure. Inverting the functor of tensoring with a specific object is an important procedure in the realm of motives and stable homotopy theory.

2 Equivalences on Algebraic Cycles (Clemens Jörder)

Introducing a type of equivalence relations on algebraic cycles, one introduces various categories of pure motives, depending on the particular equivalence relation, by taking smooth projective varieties as objects and correspondences as morphisms. These correspondences are algebraic cycles (modulo the equivalence relation) and behave to morphisms of varieties as relations behave to mappings of sets; more precisely they behave like multivalued functions. The most important equivalence relations are numerical equivalence, homological equivalence and rational equivalence, the latter introducing the ultimate necessity to have a good intersection theory.

The talk has to introduce the notion of adequate equivalence relation on cycles and the main examples with their relations to each other, including Voevodsky's smash-nilpotence relation [And04, 3.1, 3.2, p.17–22]. The facts from intersection theory that are used should be stated clearly, but without proof, cf. [Har77, Appendix A]. The conjectures about classical adequate equivalence relations and their implications should be stated.

3 Weil Cohomology Theories (Maximilian Schmidtke)

On the category of smooth projective varieties, a Weil cohomology theory is an algebro-geometric analogue of an Eilenberg-Steenrod cohomology theory (well, not quite!), i.e. there is a set of axioms and some different theories that satisfy the axioms. Betti cohomology is the Weil cohomology theory obtained from associating to a smooth variety over a field k with embedding $k \hookrightarrow \mathbb{C}$ a complex manifold and then taking the singular cohomology. This comes with a Hodge structure. In contrast, ℓ-adic cohomology is a Weil cohomology theory defined for varieties over fields of characteristics $p \neq \ell$, given by the rather complicated definition of étale cohomology, the sheaf cohomology in the étale site. This comes with the structure of a Galois representation. There is also crystalline/rigid cohomology, which plays the role of p-adic cohomology and in characteristics 0 there is also algebraic de Rham cohomology.

This talk should not try to prove anything about étale or crystalline cohomology, but introduce the idea of a Weil cohomology theory [And04, 3.3, p.23–30] while hinting at the fact that there is extra structure on the classical Weil cohomology groups (such as Hodge structure or Galois actions). The known comparison morphisms for classical Weil cohomology theories should be stated, without proof. Either Betti or algebraic de Rham cohomology can be discussed in more detail, maybe even with some proof ideas. We should see the (at least second) cohomology of $\mathbb{P}^1$ (or $\mathbb{P}^n$) in various Weil cohomology theories.

4 Pure Motives (Konrad Voelkel)

We want to have precise definitions of the category of Chow motives, homological motives and numerical motives [And04, 4.1, p.30–33]. We want to see in detail Jannsen's proof of semisimplicity of numerical motives [Jan92], which is only 6 pages long. The conjectural picture is that the numerical motives form the semisimple part of the abelian category of mixed motives and that numerical motives are the same as homological motives.

As concrete examples, we should work out the motive of projective space. This should be related to a computation of the Zeta function $Z(\mathbb{P}^n, s)$ of projective space.

5 Chow Motives of elliptic curves and Grassmannians (Sarah Kitchen)

One can completely compute the motive of generalized Grassmannians G/P for G a reductive group and P a parabolic subgroup. This talk has to explain (quickly) what a reductive group and parabolic subgroup are, and then pass to more accessible examples, like the standard Grassmannians $Gr(k, n)$ of k-planes in affine n-space.

This talk should compute motives, singular cohomology, Zeta functions and, if possible, even more, for elliptic curves and Grassmannians. The general theory of elliptic curves can be found in [Sil09], for reductive linear algebraic groups (and generalized Grassmannians) look at [Spr09], [Bor91] or [BT65].

For elliptic curves, have a look at [And04, 4.3, p.38–42]. If you already know a little bit about Fourier-Mukai transforms, you may even present the computation of the motive of a general Abelian variety.

For Grassmannians, a result from algebraic geometry, the method of Białynicki-Birula [BB73] is used, which is an algebraic analogue of Morse theory. The speaker may also give a proof of the algebro-geometric part, which is interesting in its own. How to decompose the motive, using this method, is explained in [Bro05].

6 Standard Conjectures on Algebraic Cycles (Konrad Voelkel)

This talk should introduce Grothendieck's Standard Conjectures on Algebraic Cycles as in [Gro69]. There are many conditional results, depending on the Standard Conjectures, and some of them could later be proved without assuming them, so they have been a good guiding principle in algebraic geometry.

Some relations between the Standard conjectures should be explained [And04, 5, p.47–59]. You can also take a look at [Hul94].

If the speaker has seen some rational homotopy theory before, the analogy of the Künneth type standard conjecture with formality of projective Kähler manifolds [DGMS75] can be surveyed.

If no one is found for this talk, it is merged into the next one.

2 Mixed Motives

The theory of mixed motives is explained, which for example enables us to compute motives of reductive groups (not only of their projective quotients) and affine curves (not only their projective closures). We may introduce a particular candidate for "the" category of mixed motives, discuss how this applies to number theory (periods) and prepare some preliminaries in order to understand the definition of Voevodsky's category. To have a more concrete goal in mind, one may want to read [MVW06] afterwards, but the talks should all make perfect sense without this goal. The talks on topoi and triangulated categories are very useful and interesting even outside algebraic geometry, so we may do them without introducing Voevodsky's category afterwards.

7 Mixed Hodge Structures (Michael Rottmaier & Oliver Straser)

In the realm of complex algebraic varieties, one has a mixed Hodge structure on the singular cohomology groups. That is a weight filtration whose subquotients carry a pure Hodge structure in a compatible way. One can see such a structure as a representation of the Weil restriction of GL_1 from $\mathbb{C}$ to $\mathbb{R}$, which gives rise to a $U(1)$-representation, whose Zariski-closure one calls **Mumford-Tate group**. Zariski closure is a special case of the Tannakian "dual", so the Mumford-Tate group is in spirit similar to the motivic Galois group. It is conjectured that the functor from motives of complex algebraic varieties to mixed Hodge structures (the Hodge realization) is conservative, i.e. that the motive has no more structure than the mixed Hodge structure, and that the motivic Galois group coincides with the Mumford-Tate group.

The purpose of this talk is to illustrate the "mixed philosophy" in a particular example. We should see the mixed Hodge structure of an affine curve, explained in [Del71] and [Voi02, 8.4, p.207]. If the speaker wants to do so, the situation may be compared to weight structures in l-adic cohomology of varieties over finite fields, where the survey [Jan10] on weights should be helpful. The definition of the Mumford-Tate group and the relation to the Hodge conjecture and motives is [Ara08].

8 Sites, Topoi and cd-Structures (Helene Sigloch)

A site is the natural place to do sheaf cohomology. This optional talk has to introduce Grothendieck Pretopologies (and, if time allows, Grothendieck topologies). Voevodsky's notion of completely decomposable topologies (cd-structures) will be helpful for us, since it simplifies writing down and checking the sheaf axiom in our cases. The topologies of interest are the Zariski and the étale topology, as well as the Nisnevich topology, which is finer than the Zariski topology, but not as fine as the étale topology, which lets it have miraculously good properties.

A first introduction to sheaves is in [KS06], the basics of topos theory are in [SGA72] and a "modern" treatment (but also more abstract) is in [MM92]. Levine has some lecture notes considered prerequisites for motivic homotopy theory, that contain a section on sites [Lev03]. For cd-structures consult [Voe10, Lemma 2.9]. A good comparison of étale, Nisnevich and Zariski topologies can be found in [VM99, 3, p.94]. Compare the étale topology with the analytic topology of a smooth variety over the complex numbers. Compare the Nisnevich topology with the analytic topology of a variety over the real numbers.

Most of the time will be occupied by variants of the sheaf axiom and illustrations of the particular topologies.

9 Sketch of Voevodsky's Category (Konrad Voelkel)

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