SEMINAR ON RATIONAL HOMOTOPY THEORY

Programme

Introduction

Homotopy theory is the study of topological spaces up to homotopy equivalence. The most important invariants in this subject are the homotopy groups $\pi_i(X)$. They are defined as the sets of homotopy classes of basepoint-preserving maps from a sphere S^i to the space X:

$$\pi_i(X, x_0) = [S^i, X]_*$$

For $i \geq 1$ they are indeed groups, for $i \geq 2$ even abelian groups, which carry a lot of information about the homotopy type of X. However, even for spaces which are easy to define, they can be incredibly hard to compute.

In fact, the higher homotopy groups of the spheres are still unknown!

One doesn't even see a clear pattern in the lower dimensions:

	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
S^2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
S^3		$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
S^4			$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/12$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
S^5				$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$
S^6					$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^7						$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^8							$\mathbb{Z}$	$\mathbb{Z}/2$

One observes a lot of torsion information (the finite groups $\mathbb{Z}/n\mathbb{Z}$). This suggests that one could, as a first approximation, ignore the torsion part to simplify.

Questions.

- *Given a space X, can we compute how many copies of $\mathbb{Z}$ occur in $\pi_i(X)$?*
- *Can we classify topological spaces up to the torsion part of homotopy groups?*

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This is what rational homotopy theory is all about.

Spheres have less complicated *rational* homotopy groups:

	$\pi_2 \otimes \mathbb{Q}$	$\pi_3 \otimes \mathbb{Q}$	$\pi_4 \otimes \mathbb{Q}$	$\pi_5 \otimes \mathbb{Q}$	$\pi_6 \otimes \mathbb{Q}$	$\pi_7 \otimes \mathbb{Q}$	$\pi_8 \otimes \mathbb{Q}$	$\pi_9 \otimes \mathbb{Q}$
S^2	$\mathbb{Q}$	$\mathbb{Q}$						
S^3		$\mathbb{Q}$						
S^4			$\mathbb{Q}$			$\mathbb{Q}$		
S^5				$\mathbb{Q}$				
S^6					$\mathbb{Q}$			
S^7						$\mathbb{Q}$		
S^8							$\mathbb{Q}$	

Let's make the questions more precise. The starting point is Whitehead's theorem:

Theorem. *Let $f : X \to Y$ be a continous map between two connected CW-complexes. Then f is a homotopy equivalence if and only if f is a weak homotopy equivalence, i.e. if for all $n \in \mathbb{N}$ the induced map on homotopy groups $\pi_n(X) \to \pi_n(Y)$ is an isomorphism.*

One can consider homotopy theory as the study of spaces up to weak equivalences (= maps which induce isomorphisms of homotopy groups). Since every topological space is weakly equivalent to a CW complex ("cellular approximation"), we can say in a more fancy language, that homotopy theory is the study of topological spaces with all weak equivalences inverted:

$$Hot = Top[we^{-1}]$$

(where $Top[we^{-1}]$ means we take as objects all topological spaces, as morphisms the usual morphisms *and* additionally for each weak homotopy equivalence $f : X \to Y$ a new "morphism" (not a map) in the other direction $f^{-1} : Y \to X$ such that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$, *and* additionally all morphisms you can compose out of the old and the new ones). From the viewpoint of the latter category it is easy to make precise what ignoring torsion should mean:

Definition. *A map $X \to Y$ of simply connected spaces is a* rational homotopy equivalence*, if it induces an isomorphism of $\mathbb{Q}$-vector spaces*

$$\pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}.$$

Rational homotopy theory is then the study of spaces up to rational homotopy equivalence, i.e. the study of topological spaces with rational homotopy equivalences ($\mathbb{Q}we$) inverted:

$$RatHot = Top[\mathbb{Q}we^{-1}]$$

In contrast to the usual homotopy category, **rational homotopy theory is completely algebraic**, yet still it encodes some topological information. More precisely after imposing some finiteness and connectedness conditions there is an equivalence

$$Top[\mathbb{Q}we^{-1}] \cong dgCOM[qis^{-1}]$$

between rational homotopy theory and commutative differential-graded algebras with quasi-isomorphisms inverted. This equivalence allows to compute the rational homotopy groups of a concrete space very efficiently.

One approach to get information about the rational homotopy type is to use the singular cohomology groups with rational coefficients $H^\bullet_{sing}(-;\mathbb{Q})$. While this does tell us a lot, there are still topological spaces with the same cohomology but different rational homotopy groups. One should obviously not forget the ring structure on the cohomology, given by the cup product. To get an algebraic object which captures more information about a space than just the cohomology, one can use the singular chain complex $C^\bullet_{sing}(-;\mathbb{Q})$. A disadvantage of the singular chain complex is that the product structure corresponding to the cup product is not commutative. This disadvantage is not visible for the de Rham cohomology, which is computed from the de Rham complex $\Omega^\bullet(-)$, with wedging of forms as supercommutative product structure. A key step in the machinery of rational homotopy theory is to get a commutative differential-graded algebra like the de Rham complex for arbitrary topological spaces, not just smooth manifolds.

Much can be said about *loop spaces* ΩX. Loop spaces have a product structure up to homotopy (commonly called H-space), by concatenating loops. After studying algebraic models for loop spaces by means of fiber sequences, one can derive some interesting consequences.

Another important aspect of rational homotopy theory is *formality*. Formality can be seen as a nice rational-homotopy-invariant property that a space can have, and a typical application would be to show that Kähler manifolds are always formal, while there exist certain symplectic manifolds which are not formal, hence not all symplectic manifolds are Kähler. It is still an open problem to find out which conditions on a symplectic manifold guarantee its formality.

We had to leave out many other important aspects and applications of rational homotopy theory (like a detailed treatment of Quillen's Lie models, the Lusternik-Schnirelman covering dimension, the fundamental dichotomy between rationally elliptic and hyperbolic spaces, A_∞ algebras and many more). To get some ideas on the applicability beyond the seminar, we recommend the report on the 2011 Oberwolfach Arbeitsgemeinschaft [MFO11], where we also took some ideas for talks.

Talks

We assume the participants to be familiar with singular cohomology and the fundamental group of topological spaces, as well as with the de Rham cohomology of smooth manifolds. There are 2 talks covering the necessary homotopy theory, 5 talks covering rational homotopy theory and 5 talks about applications to various subfields of mathematics.

As general reading material along the seminar we follow [MayConcise] for the basics of homotopy theory, which you may supplement with the basic textbook [Hatcher], and the monography [FHT] for the rational homotopy theory, to which [Hess06] might be a helpful introduction. Many research articles that use rational homotopy theory also contain some background material, so it is worthwile to take a look at the bibliography.

Part I: Introduction to Rational Homotopy Theory

1. **Basics of Homotopy Theory 17.4.** The purpose of the first two talks is to introduce (or in some cases recall) some relevant notions from algebraic topology. We could follow [FHT, §1,2].

Recall the definition of homotopy groups, n-connectedness and the definition of CW complexes.

Define weak homotopy equivalences and state Whiteheads theorem in the homotopy [MayConcise, 10.3, p.76] and the homology version [Hatcher, Section 4.2, p.367]. Make sure we understand the difference between

- two spaces with the same homotopy groups,
- two weakly equivalent spaces (and the notion of weak homotopy type),
- two homotopy equivalent spaces (and the notion of homotopy type).

Define Serre and Hurewicz fibrations. Explain the path space fibration [FHT, 2(b) Ex 1], and fiber bundles [FHT, 2(d) Prop. 2.6] as examples. Construct the long fiber sequence from [MayConcise, Thm. 8.6] and derive the long exact sequence of homotopy groups. This is the source for many of the long exact sequences in algebraic topology.

Define rational homotopy equivalences, the notion of rational homotopy type and rationalizations of spaces. Show us the explicit construction of the rationalization of the sphere [Hess06, 1.1]. If you're ambitious, you can explain how to rationalize CW complexes.

2. **Simplicial Techniques 24.4.** Introduce simplicial objects (in particular simplicial sets and simplicial graded algebras) after [FHT, 10(a)] or [GJ99, I.1] and compare the two different points of view: as a functor and via explicit face and boundary morphisms. Define the adjunction between the singular set functor and geometric realization [GJ99, I.2, Prop. 2.2]:

$$Top \underset{|\cdot|}{\overset{Sing}{\rightleftarrows}} sSet$$

and mention that it allows to phrase homotopy theory entirely in the language of simplicial sets. You can briefly say something about Quillen equivalence, but we won't use model categories in the rest of the seminar, so you shouldn't spend too much time on that. The notion of Kan complex (aka "extendable simplicial set") will be helpful.

You should remind us how singular cohomology of a topological space X is defined in terms of the singular set $Sing(X)$ and the singular cochain algebra $C^*_{sing}(X) = C^*(Sing(X))$ built out of $Sing(X)$.

A useful example of (co)simplicial objects are the algebraic standard simplex and its functions $k[x_0, \dots, x_n]/(\sum x_i - 1)$.

3. **Rational Homotopy Theory (2 talks) 8.5. and 15.5.** We want to give an algebraic description of the rational homotopy category. This means we look for a rational homotopy invariant that is *sharp* in the sense that two spaces are rationally homotopy equivalent iff the invariants are the same.

As sketched in the introduction, we want to assign an analogue of the de Rham complex to any topological space X, the commutative dg-algebra of polynomial differential forms $\mathcal{A}_{PL}(X)$.

The aim of these talks is to explain this and do some computations.

Introduce (simply connected) commutative dg-algebras and polynomial differential forms [Hess06, 1.20]. State that $\mathcal{A}_{PL}$ computes rational cohomology [Hess06, 1.21] and that there is an adjunction

$$sSet \underset{\langle - \rangle}{\overset{\mathcal{A}_{PL}}{\rightleftarrows}} dgCOM^{op}$$

between topological spaces and (the opposite of) the category of commutative dg-algebras. Mention that it induces an equivalence after inverting quasi-isomorphisms and restricting to simply connected objects with finitely generated cohomology.

In order to use this equivalence for concrete computation introduce (relative) (minimal) Sullivan models [FHT, 12(a) below Example 5] . The crucial property of minimal Sullivan algebras is that every quasi-isomorphism between them is an isomorphism [FHT, 12.10]. Conclude that there is a bijection between rational homotopy types and minimal Sullivan algebras [FHT, 12].

Similarly introduce homotopies between maps of Sullivan algebras [FHT, 12] and state that they are in bijection to maps in the rational homotopy category.

Explain how the rational homotopy groups can be read off from the minimal model [FHT, 13 (c) and 15.11]. You should do some simple computations - for example the rational homotopy groups of $S^n, \mathbb{CP}^n$ and $\mathbb{CP}^\infty$ are good examples. Maybe you should even remind us of $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

5. **First Geometric Applications 29.5.** In this talk, we should briefly recall Riemannian manifolds and introduce symplectic and Kähler manifolds, while we're at it. This will be relevant in later applications.

State and sketch the proof of the comparison theorem relating the de Rham complex on a smooth manifold to the complex $\mathcal{A}_{PL}$ [FHT, Thm 11.4].

Introduce the notion of formality of a dg algebra and state that it may be checked over any field extension [FHT, 12 (c), last theorem].

You can (optionally) give a short overview on the state of the art of formality of symplectic manifolds, a theory that consists mostly of counter-examples. The papers of Cavalcanti and Gompf are a good starting point for such a discussion (but not the Merkulov paper).

The bulk of the talk should consist of examples to the notion of formality after [FHT, 12(e) Example 1–3]:

- Discuss the relation between Lie-algebra cohomology and the minimal model of a Lie group.
- Introduce nilmanifolds and show that they are almost never formal.
- Introduce symmetric spaces and show that they are formal.

6. **Examples 5.6.** This talk should determine the rational homotopy type of some spaces.

(1) H-spaces are spaces with a product-up-to-homotopy structure, which includes topological groups and loop spaces. The existence of an H-space structure is a strong condition on the homotopy type.
   - Prove that the minimal model of an H-space is an exterior algebra with zero differential after [FHT, 12 (a)] or explain that the cohomology of an H-spaces is a Hopf algebra and state the classification of finite dimensional Hopf algebras over the rational numbers (cf. [MM65, Appendix]), which gives the same result.
   - Conclude that H-spaces are always formal and that Lie groups have the rational homotopy type of a wedge of odd spheres.

(2) Pushouts are an important operation in geometry, including glueing constructions as a special case.
   - Explain how models behave under pushouts [FHT, 13.5] and [FHT, 13.6] and conclude that suspensions are formal [FHT, 13.9]
   - Explain again how to read off the rational homotopy groups from the minimal model and introduce the Whitehead product (or bracket), which gives a Lie algebra structure on the rational homotopy groups. [FHT, 13(c)] Roughly speaking its role is dual to the cup product on cohomology. Explain how it is encoded in the differential of the minimal model [FHT, 13.16] This is an example of a phenomenon called Koszul duality.
   - Compute the Whitehead product in the case of spheres, projective space and H-spaces.
   - If there is still time, do some very concrete computation, for example [FHT, 13(e) Ex. 2].

7. **Loop Spaces 12.6.** This talk focuses on loop spaces and their models in rational homotopy theory.

(1) We already know that the minimal model of a loop space is an exterior algebra with vanishing differential. We would like to know the number and degrees of its generators.
   - Explain how models behave under fibrations [FHT, Thm 15.3] and [Hess06, Thm 2.2].
   - Apply this to the path space fibration to compute the minimal model of loop spaces [Hess06, Ex 2.3].

(2) Explain how models behave under pullback [FHT, 15(c)] and apply it to compute the Sullivan model of a free loop space [FHT, 15(c) Example 1]. This will be useful in talk 8.

(3) If time allows, explain the Cartan-Serre theorem about a relation of the primitive elements in the cohomology of a loop space to the rational homotopy groups [FHT, 16.10].

Part II: Applications of Rational Homotopy Theory

8. **Geodesics (Goette) 19.6.** Geodesics on Riemannian manifolds are curves which locally minimize distances, often defined in terms of a 2nd order ordinary differential equation. After quickly introducing this concept, we want to study Morse theory on the free loop space M^{S^1} over a (simply connected, compact) smooth manifold M and derive insights about geodesics on M from the rational homotopy theory of M^{S^1}.

In particular, we want to understand geodesics as critical points of the geodesic action functional on M^{S^1}, how the number of critical points is related to Betti numbers and how one can use this to show that there are infinitely many geometrically distinct closed geodesics on simply conncected, compact Riemannian manifolds which need at least two generators for their rational cohomology algebra. The geometric part of this theory was done by Gromoll and Meyer in [GM69] and the rational homotopy part by Vigué-Poirrier and Sullivan in [VPS76].

9. **String Topology (Fabert) 26.6.** In string topology, product and bracket structures on loop spaces (coming out of loop composition and “Umkehr maps”) give rise to algebras that are useful in string theory (like Gerstenhaber and Batalin-Vilkovisky algebras). A possible reference to follow is [Che12].

10. **Iterated Integrals (Huber-Klawitter) 3.7.** Chen's theory of iterated path integrals is about iterating the integration of differential forms along paths, and giving that process a geometric interpretation, which relates analysis on a manifold to the (co)homology of its path/loop space [Che77]. It can be used, for example, to put a mixed Hodge structure on the unipotent completion of the group ring of the fundamental group of any smooth complex algebraic variety. This talk could introduce (co)bar constructions. A good introduction to the subject was given by Hain [Hai01].

11. **Tate Motives and Rational Homotopy theory (Wendt) 10.7.** Tate Motives are a nice subcategory of all Motives (Motives are the universal domain of cohomology theories on schemes), which have the structure of a rigid Tannakian category, i.e. it is the representation category of an algebraic group.

In [Lev10], the differential graded modules over the algebra of cycles ($\mathbb{Z}$-linear combinations of closed subvarieties) of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ are used to describe the category of Tate motives, extending ideas of Deligne and Goncharov [DG05]. These are essentially ideas from rational homotopy theory.

12. **Formality of Kähler Manifolds (Soergel) 17.7.** We conclude the seminar with the proof of [DGMS] that every Kähler manifold is formal. We want to understand the dd^c-lemma on Kähler manifolds and how it implies formality. If time allows, we can also discuss Massey products.

There is also the interesting fact that this proof fails for generalized complex manifolds, even if they admit a dd^c-lemma [Cav07].

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