

Infinite Loop Spaces*

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This talk introduces infinite loop spaces, Ω -spectra and generalized cohomology theories and shows how these concepts relate. While infinite loop spaces are precisely the connective spectra, the main theorem in this talk is Brown representability, which states that generalized cohomology theories are in bijection to Ω -spectra.

By $*$:= $\{\text{pt}\}$ we denote the one-point space. All spaces are pointed CW complexes. All homotopies will be basepoint-preserving. The symbol \simeq means **weak homotopy equivalence**, not homotopy equivalence or homeomorphism. Some definitions and propositions are slightly more general than necessary for the theorems we prove, but in the end the model-categorical framework should simplify things by exposing what really matters.

1 Loop Spaces

Definition 1. Let X be a connected space endowed with a morphism $m : X \times X \rightarrow X$ and a basepoint $e : * \rightarrow X$. If the maps $m \circ (\text{id} \times e)$ and $m \circ (e \times \text{id})$ are both base-point preserving homotopic to e , X is called **H -space** (where the H stands for Heinz Hopf).

Example. Any topological monoid is an H -space. Note that a general H -space need not be associative, not even up to homotopy.

Definition 2. Let Y be a topological space. Then the **loop space over Y** is the topological space ΩY obtained as homotopy pullback of the basepoint inclusion along itself:

$$\begin{array}{ccc} \Omega Y & \rightarrow & * \\ \downarrow & & \downarrow \\ * & \rightarrow & Y \end{array}$$

Explicitly, this means $\Omega Y = \{\gamma \in Y^{S^1} \mid \gamma(0) = *\}$ for a CW complex Y .

Lemma 1. *The loop space ΩY of any space Y is an H -space by choosing $m : \Omega Y \times \Omega Y \rightarrow \Omega Y$ to be any way of composing maps. Furthermore, the H -space structure doesn't depend on m , up to isomorphism of H -spaces. \square*

Definition 3. Let X be a topological space. Then the **(reduced) suspension of X** is the topological space ΣX obtained as homotopy pushout of the constant map to the basepoint along itself:

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$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

Explicitly, this means $\Sigma X = S^1 \wedge X$.

Question. How to characterize which H -spaces X are loop spaces?

Lemma 2. *The functor Σ is left adjoint to Ω in the homotopy category:*

$$[\Sigma X, Y] \simeq [X, \Omega Y]$$

Proof. Follows from the basic adjunction $\text{Hom}(S^1 \times X, Y) \simeq \text{Hom}(X, Y^{S^1})$. □

Definition 4. Any space X that admits a space Y such that X is homotopy equivalent to ΩY is called a **loop space**. Such an X is called an n -fold **iterated loop space** if there are spaces Y_k for $k = 0, \dots, n$ such that $X = Y_0$ and $Y_k \simeq \Omega Y_{k+1}$ for all $k < n$. An **infinite loop space** is a space which is an n -fold loop space for all $n \in \mathbb{N}$.

Question. Which loop spaces are iterated loop spaces or even infinite loop spaces?

Example. Let π be a group, then the Eilenberg-Mac Lane space $K(\pi, n)$ is a CW complex (unique up to homotopy equivalence) such that $\pi_k(K(\pi, n))$ is π for $n = k$ and 0 otherwise. Rather trivially, given a $K(\pi, n)$ you get a $K(\pi, n - 1)$ by taking the loop space:

$$\pi_k(\Omega K(\pi, n)) = [S^k, \Omega K(\pi, n)] = [S^{k+1}, K(\pi, n)] = \begin{cases} \pi, & k = n - 1, \\ 0 & k \neq n - 1. \end{cases}$$

This shows that a $K(\pi, n)$ is homotopy equivalent to the loop space of $K(\pi, n + 1)$, thus an infinite loop space.

Question. How do infinite loop spaces arise “in nature”?

Fact. If \mathcal{C} is a strict symmetric monoidal category then the classifying space $BC = |NC|$ (the geometric realization of the nerve) is an infinite loop space. In fact, all infinite loop spaces arise from this “machine” and all such machines are equivalent in some sense (see May and Thomason: “The uniqueness of infinite loop space machines” (1978)).

2 Spectra

Remark. You might have invented spectra: Try to construct a category where suspension is invertible up to homotopy. You may motivate yourself by the fact that singular cohomology is representable in the category of spaces (by an infinite loop space), but topological K-theory is not. Try to define meaningful negative dimensional homotopy groups for the objects of such a category of spectra.

Definition 5. A **CW-spectrum** is a sequence of CW complexes E_n , indexed over the integers, together with **structure maps** $\Sigma E_n \rightarrow E_{n+1}$ (where it is allowed that all negative terms E_{-n} vanish, of course).

Remark. There are various other definitions of spectra, together with definitions for morphisms of spectra, and fibrations and cofibrations, such that there are a plethora of model categories of spectra. Most of them are Quillen adjoint and thus yield isomorphic homotopy categories. We don’t need this generality right now, so we restrict to a very restrictive class of spectra in the following.

2.1 Ω -spectra

Definition 6. An Ω -spectrum E is a set of CW complexes E_n , $n \in \mathbb{N}$, together with weak equivalences $E_n \hookrightarrow \Omega E_{n+1}$ called **structure maps**. A **map of Ω -spectra** $f : E \rightarrow F$ is a set of morphisms $(f_n : E_n \rightarrow F_n)$ such that the following diagrams commutes:

$$\begin{array}{ccc} E_n & \xrightarrow{\sim} & \Omega E_{n+1} \\ f_n \downarrow & & \downarrow \Omega f_{n+1} \\ F_n & \xrightarrow{\sim} & \Omega F_{n+1} \end{array}$$

The **category of Ω -spectra** is thus defined. Its homotopy category, given by inverting maps of Ω -spectra which are level-wise homotopy equivalences, is called the **stable homotopy category** (of CW complexes).

Remark. One can always ensure that the structure maps are homeomorphisms by taking a homotopy equivalent spectrum. We could have defined everything in a basepoint-free setting and add basepoints later, that is also equivalent.

Proposition 1. *The n -th space of an Ω -spectrum is an infinite loop space.*

Example. The **Eilenberg-Mac Lane spectrum** is given by the spaces $K(\pi, n)$. The **complex K-Theory spectrum** is given by $\mathbb{Z} \times BU$ in even degree and U in odd degree (it is an Ω -spectrum by Bott periodicity).

Definition 7. The homotopy groups of an Ω -spectrum $E = (E_n)$ are defined for integer degrees $n \in \mathbb{Z}$ as

$$\pi_n(E) := \begin{cases} \pi_0(E_{-n}), & n \leq 0 \\ \pi_n(E_0), & n \geq 0 \end{cases}$$

Proposition 2. *The homotopy groups of an Ω -spectrum are always abelian groups.*

Proof. Since $\pi_i(E_n) = \pi_{i+1}(E_{n+1})$, for all non-negative n the set $\pi_{-n}(E) = \pi_0(E_n) = \pi_2(E_{n+2})$ is an abelian group and $\pi_n(E) = \pi_n(E_0) = \pi_{n+2}(E_2)$ as well. \square

Definition 8. An Ω -spectrum E is called **connective** (or (-1) -connected), if all negative degree homotopy groups vanish, i.e. if all spaces E_n (except possibly E_0) are connected.

Example. The Eilenberg-Mac Lane spectrum is connective, since all $K(\pi, n)$ for $n \geq 1$ are connected. On the contrary, the complex K-Theory spectrum with $\mathbb{Z} \times BU$ in even degree and U in odd degree, is clearly not connective.

2.2 From spaces to spectra and the other way around

Definition 9. Given a CW complex X , one defines its **naive suspension spectrum** to be the sequence

$$X_n := \Sigma^n X$$

with structure maps the identity. By functoriality of Σ , this defines a functor Σ^∞ from CW complexes to spectra.

The (non-naive) **suspension spectrum** is the spectrum

$$X_n := \operatorname{colim}_k \Omega^k \Sigma^{n+k} X$$

with structure maps the identity. It is an Ω -spectrum (in contrast to the naive suspension spectrum) and can be thought of as a cofibrant replacement of the naive suspension spectrum in an appropriate model structure on the category of spectra.

Question. What is the suspension spectrum of $\mathbb{Z} \times BU$, the 0-th space of the K-Theory spectrum?

Proposition 3. *Suspension spectra are connective Ω -spectra.*

Proof. Let X be a CW complex and $n \in \mathbb{N}$, $n \neq 0$. By definition,

$$\pi_{-n}(\Sigma^\infty X) = \pi_0(\Sigma^n X)$$

which is a set with one element, since Σ maps spaces to connected spaces. \square

Theorem 1. The naive suspension spectrum functor Σ^∞ is left adjoint (in the category of spectra) to the functor Ω^∞ from Ω -spectra to spaces given by taking the 0-th space of a spectrum. Furthermore, Σ^∞ has as image the full subcategory of connective Ω -spectra and Ω^∞ has as image the full subcategory of infinite loop spaces.

Proof. Let X be a CW complex and E a spectrum. Since $(\Omega^\infty \circ \Sigma^\infty)(X) = (\Sigma^\infty X)_0 = X$, we can easily define a projection map $\operatorname{Hom}_{\operatorname{Spec}}(\Sigma^\infty X, E) \rightarrow \operatorname{Hom}_{\operatorname{Top}}(X, E_0)$.

By associating to a map $f : X \rightarrow E_0$ its suspensions $\Sigma^k f : \Sigma^k X \rightarrow \Sigma^k E_0$ we get a map $\operatorname{Hom}_{\operatorname{Top}}(X, E_0) \rightarrow \operatorname{Hom}_{\operatorname{Spec}}(\Sigma^\infty X, \Sigma^\infty E_0)$, which we can compose with the map induced by the inclusion $\Sigma^\infty E_0 \hookrightarrow E$ to get an inverse of the projection map.

Since every connective spectrum E can be written as $\Sigma^\infty E_0$ and every infinite loop space X arises as $(\Sigma^\infty X)_0$, the other statements are clear. \square

Remark. The 0-th space of the non-naive suspension spectrum of a space X can be very different from X itself, since it “sees” only the stable behaviour of X . For example, the Barratt-Priddy-Quillen-Theorem says

$$\Omega^\infty \Sigma^\infty S^0 \simeq B\Sigma_\infty^+ \times \mathbb{Z},$$

where Σ^∞ denotes the non-naive suspension spectrum, Σ_∞ the colimit of symmetric groups and $+$ the Quillen plus-construction.

2.3 Other categories of spectra

Fact. On the category of Ω -spectra, the smash product is not associative (only up to homotopy), so Ω -spectra do not form a symmetric monoidal category. To get rid of this defect, one needs to modify the notion of spectrum and incorporate additional structure to keep track of the braiding of the smash product. A good metaphor is that Ω -spectra are an approach with explicit coordinates (the natural numbers to index the spectrum), while other approaches are coordinate-free in some sense. Two such approaches are E_∞ ring spectra and S -modules. We won't bother for now, as their stable homotopy categories are equivalent anyway.

3 Generalized Cohomology Theories

Morally, a generalized cohomology theory is a sequence of functors that behaves just like singular cohomology H^n , but may have different values on a point (and thus on everything).

Definition 10. Let $F^n : \text{CW} \rightarrow \text{Ab}$ be a sequence of functors from the category of CW complexes to abelian groups, indexed with $n \in \mathbb{Z}$. The sequence (F^n) is called a **generalized reduced cohomology theory on CW complexes**, if the following 3 axioms hold for each n and the fourth axiom holds, too:

1. **(Weak) Homotopy:** F^n is invariant under basepoint-preserving weak homotopy equivalence, and as such factors through the pointed homotopy category of CW-complexes.
2. **Wedge Sum:** F^n maps coproducts to products, i.e.

$$F^n\left(\bigvee_{\alpha} X_{\alpha}\right) \simeq \prod_{\alpha} F^n(X_{\alpha}).$$

3. **Mayer-Vietoris:** Let $X = A \cup B$ be a union of subcomplexes which each contain the basepoint. If $a \in F^n(A)$ and $b \in F^n(B)$ restrict to the same element in $F^n(A \cap B)$, there exists an element $x \in F^n(X)$ which restricts over A and B to a and b , respectively.
4. **Exactness:** For each CW pair (X, A) there is a long exact sequence

$$\cdots \rightarrow F^n(X) \rightarrow F^n(A) \rightarrow F^n(X, A) \xrightarrow{\delta} F^{n+1}(X) \rightarrow \cdots$$

where we write $F^n(X, A)$ for $F^n(X/A)$.

Fact. From the knowledge of all $F^n(S^0)$ one can derive $F^n(S^k)$ and from the knowledge of the attaching maps of a CW complex X one can derive $F^n(X)$ for all X . So the coefficients pin down a cohomology theory.

Example. Reduced singular cohomology $\tilde{H}(\cdot; R)$, with coefficients in a ring R , is a generalized reduced cohomology theory on CW complexes (by putting $\tilde{H}^n(\cdot; R) := 0$ for all $n < 0$), with coefficients $\tilde{H}^0(S^0; R) = R$ and $\tilde{H}^n(S^0; R) = 0$ for $n \neq 0$.

Remark. The axioms “Wedge Sum” and “Mayer-Vietoris” can be read as “homotopy coproduct” and “homotopy pushout”. Given the fact that any colimit may be built out of coproducts and pushouts and from this one can get the same for homotopy colimits, one might subsume the axioms under the homotopy colimit axiom: F^n should convert homotopy colimits into limits. It is an exercise to see that this is a necessary property for a functor to be representable in the homotopy category. It is easy to see that a representable functor can never map colimits to limits, because weak equivalences might change them, so one really has to work with homotopy colimits. A sufficient definition of the homotopy colimit of a diagram D is, take the functorial cofibrant replacement of the model category (for example, CW approximation for spaces and mapping cylinder constructions to replace all maps by relative CW complexes) to get a diagram D' , then apply the classical colimit. The wedge product is the coproduct and homotopy coproduct in the category of CW complexes.

3.1 From spectra to cohomology theories

The goal of this section is to show that any Ω -spectrum E defines a generalized cohomology theory by $E^n := [\cdot, E_n]$.

Proposition 4. *Given a CW complex K , the contravariant Hom-functor $[\cdot, K]$ in the basepointed homotopy category satisfies the axioms 1–3 of [Definition 10](#).*

Proof. Homotopy Invariance is built into the definition. Given a diagram D in CW, we can choose a functorial cofibrant replacement D' and then $\text{colim } D' = \text{hocolim } D$, so we have

$$\begin{aligned} [\text{hocolim } D, K] &= [\text{colim } D', K] = \pi_0(\text{Map}(\text{colim } D', K)) \\ &= \pi_0(\lim \text{Map}(D', K)) = \lim \pi_0(\text{Map}(D', K)) = \lim [D', K] = \lim [D, K]. \end{aligned}$$

□

Remark. To state the latter in more geometric terms: The wedge sum axiom is trivial by the fact that a map $\bigvee_{\alpha} X_{\alpha} \rightarrow K$ is the same as a set of maps $X_{\alpha} \rightarrow K$. To show the Mayer-Vietoris axiom, take a CW complex X with subcomplexes A, B containing the basepoint, such that $X = A \cup B$. Let $a \in [A, K]$ and $b \in [B, K]$ such that $a|_{A \cap B} = b|_{A \cap B}$. Choose maps α, β representing a and b . We can always choose β such that $\alpha = \beta$ on $A \cap B$, since every basepoint-preserving homotopy of β over $A \cap B$ can be written as continuous map $(A \cap B) \times I \rightarrow K$ and thus extended to a map $B \times I \rightarrow K$ by the constant homotopy over $B \setminus A$. So we have continuous maps that glue together over X to a map that yields the desired homotopy class.

Definition 11. Given a morphism $f : X \rightarrow Y$ of spaces, the mapping cone of f is the space obtained as homotopy pushout of f with the identity on X :

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \dashrightarrow & C(f) \end{array}$$

Explicitly, $C(f)$ is a quotient of the mapping cylinder $((I \times X) \amalg Y)/((0, x) \sim f(x))$ by $(1, x) \sim (1, x')$.

Lemma 3. *Given a relative CW complex $f : A \hookrightarrow X$, there is a long **cofiber sequence** (also called **Puppe sequence**)*

$$A \xrightarrow{f} X \rightarrow C(f) \rightarrow \Sigma A \xrightarrow{\Sigma f} \Sigma X \rightarrow \Sigma C(f) \rightarrow \dots$$

such that the application of the covariant Hom-functor $[\cdot, K]$ into some H-space K gives a long exact sequence

$$[A, K] \xleftarrow{f^*} [X, K] \leftarrow [C(f), K] \leftarrow [\Sigma A, K] \xleftarrow{(\Sigma f)^*} [\Sigma X, K] \leftarrow [\Sigma C(f), K] \leftarrow \dots$$

Proof. The cofiber sequence is built by the procedure of subsequent mapping cone constructions:

$$A \xrightarrow{f} X \xrightarrow{g} C(f) \xrightarrow{h} C(g) \xrightarrow{f_1} C(h) \xrightarrow{g_1} C(f_1) \xrightarrow{h_1} \dots$$

where we have $C(g) \xrightarrow{f_1} C(h)$ homotopy equivalent to $\Sigma A \xrightarrow{\Sigma f} \Sigma X$ and $C(h) \xrightarrow{g_1} C(f_1)$ homotopy equivalent to $\Sigma X \xrightarrow{\Sigma g} \Sigma C(f)$, which can be seen quite explicitly by drawing pictures of the maps collapsing cones.

The loop space structure on K gives $[X, A]$ and $[X, K]$ a group structure and f^* is a group homomorphism. It suffices to check exactness of

$$[A, K] \xleftarrow{f^*} [X, K] \xleftarrow{g^*} [C(f), K].$$

Since $f^* \circ g^* = (g \circ f)^*$ and $C(f)$ is homotopy equivalent to X/A , $f^* \circ g^*$ maps homotopy classes $\alpha \in [C(f), K]$ to homotopy classes in $[A, K]$ which factor through X/A , and therefore are trivial. Given a homotopy class $\beta \in [X, K]$ such that $f^*\beta \in [A, K]$ is trivial, the homotopy can be used to extend β over $C(f)$ to yield a homotopy class in $[C(f), K]$. This establishes exactness. \square

Theorem 2. Given an Ω -spectrum $E = (E_n)$, the functors $E^n(\cdot) := [\cdot, E_n]$ for $n \geq 0$ and $E^n(\cdot) := \lim_{k \rightarrow \infty} [\Sigma^k X, E_{n+k}]$ for negative n form a generalized cohomology theory.

Proof. We have the first three axioms by [Proposition 4](#). To get the fourth axiom we use [Lemma 3](#) to get for $n \geq 0$:

$$\begin{array}{ccccccc} [X, E_{n+1}] & \rightarrow & [A, E_{n+1}] & \rightarrow & [X/A, E_{n+1}] & \rightarrow & [\Sigma X, E_{n+1}] \rightarrow [\Sigma A, E_{n+1}] \rightarrow [\Sigma X/A, E_{n+1}] \\ & & & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ & & & & [X, \Omega E_{n+1}] & \rightarrow & [A, \Omega E_{n+1}] & \rightarrow & [X/A, \Omega E_{n+1}] \\ & & & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ & & & & [X, E_n] & \longrightarrow & [A, E_n] & \longrightarrow & [X/A, E_n] \end{array}$$

For fixed negative n , we use the fact that, for any $m \geq -n$:

$$E^n(X) = \lim_{k \rightarrow \infty} [\Sigma^k X, E_{n+k}] \simeq [\Sigma^m X, E_{n+m}]$$

so we can use what we already have for positive n by considering $\Sigma^k X$ for large k instead of X . \square

Remark. The last trick amounts to “continuing the long exact sequence obtained from the cofiber sequence indefinitely to the left, too”.

Example. The Eilenberg-Mac Lane spectrum gives singular cohomology, as you can see from the coefficients

$$E^n(S^0) = [S^0, K(\pi, n)] = \pi_0(K(\pi, n)) = \begin{cases} \pi, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The K-Theory spectrum $(\mathbb{Z} \times BU, U, \dots)$ gives complex K-Theory (by the same argument), but one can observe that complex K-Theory actually has non-vanishing cohomology groups in negative degrees.

Lemma 4. *The long exact sequence axiom is equivalent to the conjunction of these two:*

a) *For each relative CW complex $i : A \hookrightarrow X$, and each $n \in \mathbb{Z}$, this sequence is exact:*

$$E^n(C(i)) \rightarrow E^n(X) \rightarrow E^n(A)$$

b) *For each CW complex X and each n , there is an isomorphism, natural in X :*

$$E^n(X) \simeq E^{n+1}(\Sigma X).$$

\square

3.2 Brown representability

In this section, we show that a generalized cohomology theory E^\bullet is the same thing as an Ω -spectrum E_\bullet , which is a theorem of Edgar Brown (1962).

Lemma 5. *Let $h : \mathcal{H}oCW \rightarrow \mathbf{Ab}$ be a functor that satisfies axioms 1–3, then*

a) *any relative CW complex $i : A \hookrightarrow X$ yields an exact sequence*

$$h(C(i)) \rightarrow h(X) \rightarrow h(A)$$

b) $h(\mathbf{pt}) = 0$

Proof. The sequence comes from functoriality of h . The inclusion $\text{Im} \subset \text{Ker}$ holds since the composition $A \hookrightarrow X \hookrightarrow C(i)$ is nullhomotopic. The inclusion $\text{Ker} \subset \text{Im}$ can be proved by decomposing $C(i) = Y \cup Z$ in two subspaces, Y a smaller copy of CA and Z a smaller copy of the mapping cylinder $M(i)$. Since the mapping cylinder deformation retracts onto X , every $x \in h(X)$ extends to $z \in h(Z)$ and if x restricts to the trivial element of $h(A)$, z restricts to the trivial element of $h(Y \cap Z)$. By Mayer-Vietoris this yields an element of $h(C(i))$ that restricts to $z \in h(Z)$ and hence to $x \in h(X)$.

For the second claim: The inclusion map $X \hookrightarrow X \vee \mathbf{pt}$ induces the projection map $h(X) \times h(\mathbf{pt}) \rightarrow h(X)$, which can be composed with the isomorphism $h(X) = h(X \vee \mathbf{pt}) \simeq h(X) \times h(\mathbf{pt})$ to yield the identity $h(X) \xrightarrow{\text{id}_{h(X)}} h(X)$, so the projection $h(X) \times h(\mathbf{pt}) \rightarrow h(X)$ must be an isomorphism, too. \square

Lemma 6. *From [Lemma 4](#) we know that the exactness axiom can be replaced, given the other axioms, by natural isomorphisms*

$$E^n(X) \simeq E^{n+1}(\Sigma X)$$

Theorem 3 (Brown Representability). Every (reduced) generalized cohomology theory (on base-pointed CW complexes) E^n is represented by an Ω -spectrum E_n which is unique up to homotopy equivalence:

$$E^n(X) \simeq [X, E_n].$$

The proof (following Hatcher) consists of constructing n -universal pairs for each E^k by attaching cells inductively, to show that this yields a π_* -universal pair in the homotopy limit, to prove that a π_* -universal pair is universal and finally to stick the representing spaces together to a spectrum representing E .

Definition 12. Let h be a functor that satisfies axioms 1–3 of a generalized cohomology theory, i.e. a functor $h : \mathcal{H}oCW \rightarrow \mathbf{Ab}$ that maps homotopy colimits to limits. A pair (K, u) consisting of a CW complex K and a class $u \in h(K)$ are said to be

- **universal**, if for every CW complex X , the pullback $[X, K] \xrightarrow{(\cdot)^*(u)} h(X)$ is an isomorphism;
- **n -universal**, if the pullback $\pi_i(K) \xrightarrow{(\cdot)^*(u)} h(S^i)$ is surjective for $i \leq n$ and has trivial kernel for $i < n$;
- **π_* -universal**, if it is n -universal for all $n \in \mathbb{N}$.

Remark. Admitting a universal pair is equivalent to being representable: if h is representable by K we can define u to be the image of the identity id_K under $[K, K] \xrightarrow{\sim} h(K)$. The data of a universal class u is therefore equivalent to a chosen natural transformation $[\cdot, K] \xrightarrow{\sim} h$.

Lemma 7. *Given any pair (Z, z) with Z a connected CW complex and $z \in h(Z)$, there exists an n -universal pair (K_n, u) for h with Z a subcomplex of K_n and $u|_Z = z$.*

Proof. We construct K_n inductively from Z by attaching cells. Let

$$K_1 := Z \vee \bigvee_{\alpha \in h(S^1)} S^1_\alpha,$$

then K_1 is connected and $h(K_1) = h(Z) \times \prod_{\alpha \in h(S^1)} h(S^1)$ and we can choose $u_1 \in h(K_1)$ to be $u_1|_Z = z$ and $u_1|_{S^1_\alpha} = \alpha$. The pair (K_1, u_1) is 1-universal, since $0 = \pi_0(K_1) \xrightarrow{u_1^*} h(S^0)$ has trivial kernel and $\pi_1(K_1) \xrightarrow{u_1^*} h(S^1)$ is surjective by construction.

For the inductive step, represent each element $\alpha \in \text{Ker}(\pi_n(K_n) \rightarrow h(S^n))$ by a map $f_\alpha : S^n \rightarrow K^n$ and let $f := \bigvee_\alpha f_\alpha : \bigvee_\alpha S^n_\alpha \rightarrow K_n$. The mapping cone $C(f)$ is obtained from K_n by attaching $(n+1)$ -cells along the f_α and we define

$$K_{n+1} := C(f) \vee \bigvee_{\beta \in h(S^{n+1})} S^{n+1}_\beta.$$

The mapping cylinder $M(f)$, of which $C(f)$ is the quotient by $\bigvee_\alpha S^n_\alpha$, deformation retracts to K_n , so $u_n \in h(K_n) \simeq h(M(f))$. By the definition of f , this $u_n \in h(M(f))$ restricts to $u_n|_{\bigvee_\alpha S^n_\alpha} = 0$, since f was built this way. The exactness property of h ([Lemma 5](#)) gives an element $w \in h(C(f))$ which restricts to u_n on K_n . From $h(K_{n+1}) = h(C(f)) \times \prod_{\beta \in h(S^{n+1})} h(S^{n+1})$ we see that w extends to $u_{n+1} \in h(K_{n+1})$ such that $u_{n+1}|_{C(f)} = w$ and $u_{n+1}|_{S^{n+1}_\beta} = \beta$. The pair (K_{n+1}, u_{n+1}) is now n -universal since homotopy groups in dimension $\leq n$ depend only on the n -skeleton, which we didn't change. It is also $(n+1)$ -universal, as we can obtain from the following commutative diagram:

$$\begin{array}{ccc} \pi_i(K_n) & \rightarrow & \pi_i(K_{n+1}) \\ & \searrow & \swarrow \\ & h(S^i) & \end{array}$$

Since K_{n+1} is obtained from K_n by attaching $(n+1)$ -cells, the upper map is an isomorphism for $i < n$ and a surjection for $i = n$. From the induction we know that the left map has trivial kernel for $i < n$ and is surjective for $i \leq n$. The kernel of the right map is trivial for $i = n$, since any element of the kernel would come from $\pi_i(K_n)$ by surjectivity of the upper map, would be in the kernel of the left map by commutativity, and thus has to be killed by the upper map, since we attached cells to K_n that kill all of the kernel of the left map, to obtain K_{n+1} . For $i = n+1$, the right map is surjective by construction. \square

Question. This was one case, where an injectivity condition follows from some surjectivity one level below. Can you formalize when such a pattern occurs?

Lemma 8. *Given any pair (Z, z) with Z a connected CW complex and $z \in h(Z)$, there exists a π_* -universal pair (K, u) for h with Z a subcomplex of K and $u|_Z = z$.*

Proof. Let $K := \bigcup_n K_n$, this gives the right space. To obtain the class u , we refine this description at the homotopy level by the **mapping telescope**, a certain homotopy colimit, of $K_1 \hookrightarrow K_2 \hookrightarrow \dots$. It is given explicitly by the construction

$$T := \bigcup_i K_i \times [i, i+1] \subset K \times [1, \infty)$$

and the projection $T \rightarrow K$ is a homotopy equivalence, since $K \times [1, \infty)$ deformation retracts onto T . From the property that h maps homotopy colimits to limits we can write $h(K) \simeq h(T) = \lim_n h(K_n)$ and the n -universal classes u_n correspond to a class $u \in h(K)$. The pair (K, u) is π_* -universal, as one can deduce from the following diagram:

$$\begin{array}{ccc} \pi_i(K_n) & \longrightarrow & \pi_i(K) \\ & \searrow & \swarrow \\ & h(S^i) & \end{array}$$

For $n > i + 1$, the upper map is an isomorphism and the left map is surjective with trivial kernel, and so must be the right map. \square

Lemma 9. *Let (K, u) be a π_* -universal pair for h and (X, A) a CW pair. Then for each $x \in h(X)$ and each map $f : A \rightarrow K$ with $f^*(u) = x|_A$ there exists a map $g : X \rightarrow K$ extending f with $g^*(u) = x$.*

Proof. Replacing K by the mapping cylinder of f , we can take f to be the inclusion of a subcomplex. Let $Z := X \cup_A K$. By Mayer-Vietoris, there is $z \in h(Z)$ that restricts to $x \in h(X)$ and $u \in h(K)$. From **Lemma 8** we can embed (Z, z) in a π_* -universal pair (K', u') . The inclusion $(K, u) \hookrightarrow (K', u')$ induces an isomorphism on homotopy groups since both are π_* -universal, so K' deformation retracts onto K . This gives a homotopy relative A of the inclusion $X \hookrightarrow K'$ to some map $g : X \rightarrow K$. We have $g^*(u) = x$ since u' restricts to u over K and to x over X . \square

Proposition 5. *Let $h : \mathcal{H}oCW \rightarrow \mathbf{Ab}$ be a functor that maps homotopy colimits to limits. Then a π_* -universal pair (K, u) for a functor h is universal, hence by **Lemma 8** there exists a universal pair (K, u) .*

Proof. From **Lemma 9** applied to $A = \text{pt}$ we get a surjection $[X, K] \twoheadrightarrow h(X)$. For injectivity we apply **Lemma 9** to $((X \times I)/(* \times I), X \times \partial I)$: If $f_0^*(u) = f_1^*(u)$, take f to be f_0, f_1 on $X \times \partial I$ and $x := \text{proj}_X^* f_0^*(u) = \text{proj}_X^* f_1^*(u)$. The lemma gives a homotopy from f_0 to f_1 .

Alternatively, we can proceed by induction over the cells to get for every finite CW complex $Z \cup_f e^n$ such that $[Z, K] \xrightarrow{\sim} h(Z)$ an isomorphism $[Z \cup_f e^n] \xrightarrow{\sim} h(Z \cup_f e^n)$ from the long exact sequences for $[\cdot, K]$ and h obtained from the cofiber sequence $Z \hookrightarrow Z \cup_f e^n \rightarrow S^{n+1}$, by applying the 5-lemma. \square

Proof of Theorem 3. Let E^n be a generalized cohomology theory. From **Proposition 5** we have universal pairs (K_n, u_n) for each E^n . It remains to show that the natural isomorphisms $E^n(X) \simeq E^{n+1}(\Sigma X)$ correspond to weak equivalences $K_n \rightarrow \Omega K_{n+1}$. At least they correspond to natural isomorphisms

$$[X, K_n] \simeq E^n(X) \simeq E^{n+1}(\Sigma X) \simeq [\Sigma X, K_{n+1}] \simeq [X, \Omega K_{n+1}]$$

which we call Φ . From naturality we get for each map $f : X \rightarrow K_n$ a commutative diagram

$$\begin{array}{ccc}
[K_n, K_n] & \xrightarrow{f^*} & [X, K_n] \\
\Phi \downarrow & & \downarrow \Phi \\
[K_n, \Omega K_{n+1}] & \xrightarrow{f^*} & [X, \Omega K_{n+1}]
\end{array}$$

We have $\varepsilon_n := \Phi(1) : K_n \rightarrow \Omega K_{n+1}$ as candidate for the weak equivalence we need. From the diagram we see that $\Phi(f) = f^* \varepsilon_n$, so Φ is just composition with ε_n , so taking $X = S^i$ and remembering Φ is a bijection shows that ε_n induces isomorphisms on π_i .

Regarding the group structure on $E^n(X)$ and the group structure on $[X, K_n] \simeq [\Sigma X, K_{n+1}]$, we can identify them since they distribute over each other: For maps $f, g : \Sigma X \rightarrow K_{n+1}$, we have $(f + g)^* = f^* + g^* : h(K_{n+1}) \rightarrow h(\Sigma X)$ (which is proved by pulling back along $f \vee g$). \square

Remark. If we define the internal mapping object in the category of Ω -spectra, for two Ω -spectra F, E to be $[F, E]_n := [F, \Sigma^n E]$, then for any CW complex X we have

$$[X, E_n] \simeq [\Sigma^\infty X, E]_n = [\Sigma^\infty X, \Sigma^n E]$$

since the maps of spectra $\Sigma^\infty X \rightarrow E$ are uniquely determined by their 0-th term and can be recovered from them, up to homotopy. This gives us the possibility to “represent” a generalized cohomology theory E^\bullet in the category of Ω -spectra by

$$[\Sigma^\infty X, E]_\bullet \xrightarrow{\sim} E^\bullet(X).$$

By extending every generalized cohomology theory to a functor on the stable homotopy category, we can even say that we have a bijection

$$[F, E]_\bullet \xrightarrow{\sim} E^\bullet(F) \text{ for all spectra } F.$$

Question. How much of Brown’s representability theorem carries over to a general (pointed, simplicial?) model category?

Fact. Jardine proved in “Representability theorems for presheaves of spectra” (2011) that an analogue of Brown representability holds for simplicial presheaf categories over pointed (co)complete cofibrantly generated simplicial model categories that satisfy certain conditions (similar to those holding for spectra).

On the other hand, there are examples of model categories for which Brown representability fails.

3.3 Spectra and Homology

Remark. Similar axioms can be given for generalized homology and similar remarks apply to singular homology.

One can also define a generalized homology theory E_\bullet from an Ω -spectrum E by

$$E_n(X) := \pi_n(X \wedge E),$$

where $X \wedge E$ is the spectrum obtained from E by smashing each space E_n with X .

Fact. Given an Ω -spectrum $E = (E_n)$, the homology theory E_n defined by it has vanishing negative degree groups if and only if E is connective.

Remark. The corresponding theorem for cohomology is wrong, despite the fact that singular cohomology makes it look like it was true. The cause for singular cohomology to have no negative degree cohomology groups is different, though: the homotopy groups of Eilenberg-Mac Lane spaces $K(\pi, n)$ **above** n vanish.

3.4 Shifts in the category of spectra

Definition 13. On the level of spectra, one can define a suspension functor Σ , which maps a spectrum $E = (E_n)$ to the spectrum ΣE with $(\Sigma E)_n = \Sigma(E_n)$. One can also **formally desuspend** by defining $\Sigma^{-1}E$ to be the spectrum with $(\Sigma^{-1}E)_n = \Omega(E_n)$.

Proposition 6. *Suspension and formal desuspension are inverse up to weak homotopy, so they give inverse functors on the stable homotopy category.*

Proof. Explicitly, given a spectrum E , we compute

$$(\Sigma^{-1}\Sigma E)_n = \Omega\Sigma(E_n) \text{ and } (\Sigma\Sigma^{-1}E)_n = \Sigma\Omega(E_n),$$

and conclude by using $E_{n-1} \simeq \Omega E_n$ as well as $\Sigma E_{n-1} \simeq E_n$. □

Theorem 4 (Vogt). The suspension as translation and the long sequences obtained by subsequent mapping cone constructions

$$X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{j} C(i) \simeq \Sigma X \xrightarrow{\Sigma f} \Sigma Y \simeq C(j) \rightarrow \dots$$

define the structure of a triangulated category on the stable homotopy category.

Fact. The monoidal structure given by the smash product, is compatible, on the stable homotopy category, with the triangulated structure, and associative and commutative (**not** on the category of spectra, as defined above).