Homotopy Sheaves and h-Principles

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We introduce Grothendieck topologies and model categories along familiar examples, proceed by defining the Jardine model structure on simplicial presheaves and then relate homotopy sheaves to h-principles such as Gromov's Oka principle.

1 Grothendieck Topologies and Sheaves

The following definition should be more-or-less familiar:

Definition 1. For X a topological space, let $Ouv(X)$ be the category of open subsets of X, with inclusions as morphisms. Then a **presheaf** of abelian groups $\mathcal{F}$ on X is a contravariant functor $\mathcal{F} : Ouv(X) \to \mathcal{A}b$. A **sheaf** of abelian groups $\mathcal{F}$ on X is a presheaf that satisfies the unique glueing axiom: for all coverings $(U_i)_{i\in I}$ of open sets $U = \bigcup_{i\in I} U_i$ in $Ouv(X)$,

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram (of abelian groups), i.e. the first term is the kernel of the difference between the two right maps.

Most people also include the axiom $\mathcal{F}(\emptyset) = 0$.

Now we replace $Ouv(X)$ by a general category $\mathcal{C}$, so we have to supply a notion of "coverings" in this setting.

Definition 2. For $\mathcal{C}$ a category, a **Grothendieck pretopology** on $\mathcal{C}$ is the assignment of a collection of **coverings** to each object $U \in \mathcal{C}$, where a covering is a set of morphisms $\{U_i \to U\}$, such that the following conditions are satisfied:

1. If $f : V \to U$ is an isomorphism, then the singleton set $\{f\}$ is a covering of U.

2. If $\{U_i \to U\}$ is a covering of U and $V \to U$ any morphism, then $\{U_i \times_U V \to V\}$ is a covering of V. In particular, these fiber products are all required to exist. (Remark: The fiber product in $Ouv(X)$ is the intersection of sets.)

3. If $\{U_i \to U\}$ is a covering of U and for each i we have a covering $\{V_{ij} \to U_i\}$ of U_i, then the composites $\{V_{ij} \to U_i \to U\}$ together form a covering of U again.

Definition 3. Let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology and $\mathcal{D}$ any other category. Then a contravariant functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called a **presheaf**, and a presheaf $\mathcal{F}$ is called a **sheaf** if the glueing axiom is satisfied: For all coverings $\{U_i \to U\}$ of objects $U \in \mathcal{C}$,

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer diagram. This is called **descent along the covering family**.
...and if $\mathcal{C}, \mathcal{D}$ have initial objects $\emptyset, 0$, one also wants to have $\mathcal{F}(\emptyset) = 0$.

Any Grothendieck pretopology yields a notion of sheaves, and different pretopologies may lead to the same notion of sheaves. To have a notion of sheaves may be called a topology. A category together with a fixed Grothendieck topology is called a **site**.

Example 1. On $\mathcal{C} = Top$, let the coverings be jointly surjective collections of open continuous injective maps. This is a "big version" of the category $Ouv(X)$. It is called the global classical topology.

On $\mathcal{C} = Top$, let the covering be jointly surjective local homeomorphisms. This is called the global étale topology for topological spaces.

The sheaves for these two pretopologies coincide, i.e. they generate the same topology.

Remark 1. The forgetful functor $Shv(\mathcal{C}) \to PShv(\mathcal{C})$ has a left adjoint called **sheafification**.

1.1 Sieves

It is sometimes technically necessary to use sieves instead of coverings, for example if not all fiber products exist.

Definition 4. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$ an object. A **sieve** S over X is a full subcategory of $(\mathcal{C} \downarrow X)$ closed under precomposition with morphisms from $\mathcal{C}$, i.e. for any $f : Y \to X$ in S and any $g : W \to Y$ in $\mathcal{C}$, $f \circ g \in S$ again.

A **Grothendieck topology** on $\mathcal{C}$ is now an assignment of a set of covering sieves to each object, such that the following axioms hold:

- Pullbacks of covering sieves are again covering sieves.
- The maximal sieve $\mathrm{id} : Hom(-, X) \to Hom(-, X)$ is always covering X.
- Two sieves cover an object iff their intersection covers that object.
- If F is a sieve such that $\bigcup_Y \{g : Y \to X \mid g^* F \text{ covers } Y\}$ covers X, then F itself is covering X.

Given a Grothendieck pretopology we assign to each covering set of morphisms $\{U_i \to X\}$ the sieve generated by these morphisms, i.e. the smallest full subcategory of $(\mathcal{C} \downarrow X)$ which is closed under precomposition with morphisms from $\mathcal{C}$.

A presheaf $\mathcal{F}$ is called a **sheaf** with respect to a Grothendieck topology if for all objects U and all covering sieves S of U:

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} PShv(y(U_i) \times_{y(U)} y(U_j), \mathcal{F})$$

where y is the Yoneda embedding and $U_i \to U$ are morphisms that generate the sieve.

2 Simplicial Objects

Definition 5. Let Δ be the category of finite ordinal numbers with monotone maps as morphisms. The morphisms are generated by two classes of morphisms, the coface (leave one out) and the codegeneracy (take one twice) maps. The relations between these generating maps are called cosimplicial identities. Presheaves with values in a category $\mathcal{C}$ on the category Δ are called **simplicial objects** in $\mathcal{C}$. The cosimplicial identities translate into simplicial identities in $\mathcal{C}$. The data of a simplicial object $X_\bullet$ in $\mathcal{C}$ are equivalent to giving for each $n \in \mathbb{Z}_{\geq 0}$ an object X_n and morphisms ∂_k,s_k that satisfy the simplicial identities:

1. $\partial_i \partial j = \partial_{j-1} \partial_i$ for $i < j$,
2. $s_i s_j = s_{j+1} s_i$ for $i \leq j$,
3. $\partial_i s_j = \begin{cases} s_{j-1}\partial_i & \text{for } i < j, \\ \mathrm{id} = \partial_{j+1} s_i & \text{for } i = j, \\ s_j \partial_{i-1} & \text{for } i > j+1 \end{cases}$

Definition 6. Let $\mathcal{C}$ be a category of spaces that contains a standard n-simplex Δ^n for every n, i.e. a cosimplicial object $\Delta^\bullet$. This can be used to form $\mathcal{C}(\Delta^\bullet, \cdot) : \mathcal{C} \to sSet$, a functor that assigns to each space its **singular complex**. One can also use it to define a geometric realization into $\mathcal{C}$ by gluing together copies of Δ^n for each n-simplex of a given simplicial set.

The functors are adjoint, in fact this is $(\cdot)^{\Delta^\bullet} \dashv \Delta^\bullet \otimes (\cdot)$.

3 Model Categories

Definition 7. A model category is a category $\mathcal{M}$ with three classes of morphisms $W, C, F \subset \operatorname{Mor} \mathcal{M}$ (called weak equivalences, cofibrations, fibrations) that satisfy the following axioms:

M1 The category $\mathcal{M}$ contains all small limits and colimits.

M2 For $f, g \in \operatorname{Mor} \mathcal{M}$ such that $g \circ f$ is defined, whenever two out of $\{f, g, g \circ f\}$ are weak equivalences, so is the third.

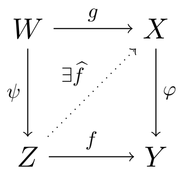
M3 If f is a retract of $g \in \operatorname{Mor} \mathcal{M}$ then f inherits the properties W, C, F of g.

M4 Cofibrations have the right lifting property wrt. fibrations that are weak equivalences. Fibrations have the left lifting property wrt. cofibrations that are weak equivalences.

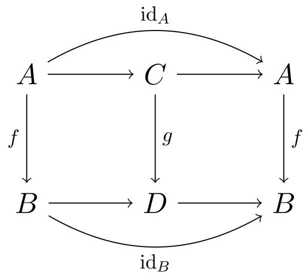
M5 For any $h \in \operatorname{Mor} \mathcal{M}$ there are two functorial factorizations $h = g \circ f$ in a fibration g and a cofibration f, in one case g is a weak equivalence in addition (cofibrant replacement), in the other case f is a weak equivalence (fibrant replacement).

This is definition 7.1.3 in Hirschhorn's book.
To remind you of lifting properties:

Definition 8. For φ, ψ in the diagram we define: whenever for all f, g the morphism $\widehat{f}$ exists (such that the diagram commutes), we say that φ has the left lifting property with respect to ψ, and conversely ψ has the right lifting property with respect to φ.



Definition 9. If there is a commutative diagram



we call f a **retract** of g.

That is definition 7.1.1 in Hirschhorn's book.

Definition 10. In a model category, X is called **fibrant**, if $X \to \mathsf{pt}$ is a fibration. X is called **cofibrant**, if $\emptyset \to X$ is a cofibration.

Remark 2. From axiom M5 one can factor (functorially) the morphisms $X \to \mathsf{pt}$ and $\emptyset \to X$ such that $X \to \widetilde{X}$ is a weak equivalence to a fibrant object, or $\widehat{X} \to X$ is a weak equivalence to a cofibrant object. These are called the functorial fibrant or cofibrant replacements of X.

Definition 11. If $\mathcal{M}$ is a model category with weak equivalences W, the localization category $[W^{-1}]\mathcal{M}$ exists and is called the **homotopy category** $\mathcal{H}o(\mathcal{M})$.

3.1 Examples

Example 2. Topological spaces admit various model structures. Here is one, the **Quillen model structure on topological spaces**:

- Weak equivalences are the weak homotopy equivalences (morphisms that induce isomorphisms on all homotopy groups).
- Fibrations are the **Serre fibrations**, i.e. maps which have the right lifting property wrt. all inclusions $D^n \xrightarrow{\simeq} D^n \times \{0\} \hookrightarrow D^n \times I$.
- Cofibrations are defined by the left lifting property with respect to fibrations that are weak equivalences. It turns out that these are then "retracts of relative cell complexes", generated by the boundary inclusions $S^{n-1} \hookrightarrow D^n$.

1. One can construct products, coproducts, equalizers and coequalizers by hand, so they exist.
2. 2-out-of-3 holds for isomorphisms, hence for morphisms inducing isos on π_*.
3. Applying π_* to a retraction diagram between two arrows shows that the property holds for weak equivalences. Since fibrations and cofibrations are defined by lifting properties, the retract axiom is easily checked for them.

4. This needs the original version of the small object argument (without further comment).
5. Functorial factorization can be done by mapping cylinders for the cofibrant factorization. For fibrant factorization, one can factor over a path space.

Example 3. Simplicial sets also admit a model structure which is called the **Quillen model structure on simplicial sets**:

- Weak equivalences are morphisms whose geometric realization is a weak homotopy equivalence of topological spaces.
- Fibrations are **Kan fibrations**, i.e. maps which have the right lifting property wrt. all horn inclusions.
- Cofibrations are monomorphisms, i.e. levelwise injective maps.

Since Set is a complete and co-complete category and simplicial sets are a category of presheaves on the simplex category Δ, simplicial sets also form a complete and co-complete category.

Remark 3. There exist adjoint functors $|\cdot| : SimpSet \to Top : Sing_\bullet$ (geometric realization and singular complex) which respect the model structure, if one puts the Quillen structure on both categories. Such a setup is called a Quillen adjunction. Because of this Quillen adjunction, one can often choose to replace simplicial sets by topological spaces or the other way around.

3.1.1 Mixed Model Structures

Theorem 1. If (W_h, C_h, F_h) and (C_q, W_q, F_q) are model structures on the same category $\mathcal{C}$ (think of Hurewicz and Quillen model structure on topological spaces), and we have inclusions $F_h \subseteq F_q$ and $W_h \subseteq W_q$, then there exists a class C_m such that (C_m, W_q, F_h) is a model structure.

In the case of topological spaces, this yields the mixed model structure, with weak homotopy equivalences, Hurewicz fibrations and the cofibrant objects are precisely those who are homotopy equivalent to a CW complex.

4 Simplicial Model Categories

Definition 12. For convenience, a simplicial category is defined as category enriched over simplicial sets (that satisfies extra axioms), but one uses the plain old Hom (or *hom*) for the 0-simplices of the enriched Hom (or Map). A simplicial model category is a simplicial category with model structure that is powered and copowered in simplicial sets (sometimes called axiom SM6), such that the compatibility axiom (SM7) holds. The compatibility axiom is a special case of "enriching a model category in a closed monoidal model category".

Theorem 2 (Fundamental theorem of model categories). Let X, Y be objects in a simplicial model category $\mathcal{M}$. Let $\widehat{X}$ be a fibrant-cofibrant replacement of X and $\widetilde{Y}$ a fibrant-cofibrant replacement of Y, then $\mathcal{M}(\widehat{X}, \widetilde{Y})$ is a simplicial set, and quotienting out the left homotopy relation yields a set isomorphic to $\mathcal{H}o(X, Y)$.

Theorem 3 (Verdier hypercovering theorem). Let $f : X \to Y$ be a morphism in $\mathcal{H}o(\mathcal{M})$, the homotopy category of the simplicial model category $\mathcal{M}$ of simplicial presheaves on a site with the global injective model structure. Then, there is a hypercover $X_\bullet \to X$ that is a weak equivalence and there is a morphism $X_\bullet \to Y$ realizing f.

In the special case $f : X \to K(\mathbb{Z}, n)$ we get the statement that cohomology $H^n(X; \mathbb{Z}) = [X, K(\mathbb{Z}, n)]$ can be calculated by hypercovers (Verdier's original version).

5 Reedy Model Categories

Definition 13. A **Reedy category** is a small category $\mathcal{C}$ together with two lluf subcategories, the raising $\overrightarrow{\mathcal{C}}$ and the lowering $\overleftarrow{\mathcal{C}}$ morphisms, such that there exists a degree map from the object set to the non-negative integers and the following axioms are satisfied: the raising morphisms raise the degree, the lowering morphisms lower the degree, and every morphism can be uniquely factored into a lowering followed by a raising morphism.

For a Reedy category $\mathcal{C}$ and an object $\alpha \in \mathcal{C}$ we define the **latching category** $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ to be the full subcategory of $(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ of all objects except id_α, similarly the **matching category** $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ the full subcategory without id_α.

In this situation, for a model category $\mathcal{M}$ and an object $X \in \mathcal{M}$ we define

$$L_\alpha X := \mathit{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} X \to X_\alpha \text{ the latching object with the latching morphism,}$$

$$X_\alpha \to M_\alpha X := \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} X \text{ the matching morphism to the matching object.}$$

For any morphism of diagrams $f \in \mathcal{M}^{\mathcal{C}}$ we define

$$X_\alpha \coprod_{L_\alpha X} L_\alpha Y \to Y_\alpha \text{ the relative latching map of } f \text{ at } \alpha,$$

$$X_\alpha \to Y_\alpha \coprod_{M_\alpha Y} M_\alpha X \to X_\alpha \text{ the relative matching map of } f \text{ at } \alpha,$$

and from these notions we define a model structure on $\mathcal{M}^{\mathcal{C}}$:

- Weak equivalences are defined object-wise.
- A map is a Reedy cofibration if at all $\alpha \in \mathcal{C}$ the relative latching map is a cofibration in $\mathcal{M}$.
- A map is a Reedy fibration if at all $\alpha \in \mathcal{C}$ the relative matching map is a fibration in $\mathcal{M}$.

6 Model Structure on Simplicial Presheaves

Definition 14. Let $\mathcal{C}$ be a site. A simplicial (pre)sheaf is a (pre)sheaf with values in simplicial sets.

On the simplicial presheaves on $\mathcal{C}$ we define a model structure, called **Heller model structure**, that doesn't incorporate the structure of the site yet: The weak equivalences and cofibrations are defined object-wise, which means that a morphism $f : \mathcal{F} \to \mathcal{G}$ of simplicial presheaves is a weak equivalence (resp. cofibration) if and only if for all $X \in \mathcal{C}$ the morphism $f_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is a weak equivalence (resp. cofibration) of simplicial sets (where we use the Quillen model structure on simplicial sets). The fibrations are defined by the right lifting property with respect to acyclic cofibrations. This is often called a **global injective model structure**. The dual **global projective model structure** would consist of defining fibrations object-wise and cofibrations via the lifting property.

The word "global model structure" indicates that the local structure of the site isn't incorporated, or that no process of localization has happened. This process of localization occupies us in the following.

There are also various **intermediate** model structures, between the injective and projective model structures, both the global and the local ones.

6.1 General theory of localization

Definition 15. Let $\mathcal{D}$ be a class of maps in a model category $\mathcal{M}$. An object W is called **$\mathcal{D}$-local** if W is fibrant and for every $(f : A \to B) \in \mathcal{D}$ the induced map of homotopy function complexes

$$f^* : map(B, W) \to map(A, W)$$

is a weak equivalence.

A map $g : X \to Y$ is called a **$\mathcal{D}$-local equivalence** if for every $\mathcal{D}$-local object W the induced map of homotopy function complexes

$$g^* : map(Y, W) \to map(X, W)$$

is a weak equivalence.

Remark: weak equivalences are $\mathcal{D}$-local equivalences for any class $\mathcal{D}$.

As some kind of reminder...

Definition 16. A (left) homotopy function complex $map(X, Y)$ of objects X, Y in a model category $\mathcal{M}$ is a triple $(\widetilde{X}, \widehat{Y}, \mathcal{M}(\widetilde{X}, \widehat{Y}))$, where $\widetilde{X}$ is a cosimplicial resolution of X (a cofibrant replacement of X in M^{Δ} with the Reedy model structure), $\widehat{Y}$ a fibrant replacement of Y and $\mathcal{M}(\widetilde{X}, \widehat{Y})$ the simplicial set given by the cosimplicial structure of $\widetilde{X}$ and the contravariant Hom-functor $\mathcal{M}(-, \widehat{Y})$ of $\mathcal{M}$.

Definition 17. Let $\mathcal{D}$ be a class of maps in a model category $\mathcal{M}$. Then the **left Bousfield localization** of $\mathcal{M}$ at $\mathcal{D}$ is (if it exists) the model structure $L_{\mathcal{D}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ that consists of

- Weak equivalences are the $\mathcal{D}$-local equivalences of $\mathcal{M}$.
- Cofibrations are the same as in $\mathcal{M}$.
- Fibrations are defined by the right lifting property.

Symmetrically, one can keep fibrations fixed and define cofibrations by the lifting property, to get the **right Bousfield localization**.

6.2 Localization of simplicial model categories

The following definition is equivalent to the more general one:

Definition 18. Let $\mathcal{D}$ be a class of maps in a simplicial model category $\mathcal{M}$.

An object W is called **$\mathcal{D}$-local** if W is fibrant and for every $(f : A \to B) \in \mathcal{D}$ the induced map of simplicial sets

$$f^* : map(\widehat{B}, W) \to map(\widehat{A}, W)$$

is a weak equivalence, where $\widehat{A}, \widehat{B}$ are cofibrant replacements in $\mathcal{M}$.

A map $g : X \to Y$ is called a **$\mathcal{D}$-local equivalence** if for every $\mathcal{D}$-local object W the induced map of simplicial sets

$$g^* : map(\widehat{Y}, W) \to map(\widehat{X}, W)$$

is a weak equivalence, where $\widehat{X}, \widehat{Y}$ are cofibrant replacements in $\mathcal{M}$.

Definition 19. We define the **fine** aka **standard** aka **Jardine** aka **left local injective model structure** on simplicial presheaves as the left Bousfield localization of the global injective model structure along the hypercoverings of the site, i.e. the morphisms $hocolim X_\bullet \to X$ where X is an object of the site, considered as presheaf $Hom(-, X)$, considered as simplicial presheaf concentrated in degree 0, and $X_\bullet$ is a hypercovering of X, which is a simplicial object in the site, considered as simplicial presheaf $Hom(-, X_\bullet)$. Alternatively, one can say that the weak equivalences and the cofibrations are defined stalk-wise, the fibrations via a lifting property – although the cofibrations are still just monomorphisms, i.e. simplicial-level-wise injections.

The characteristic of this left Bousfield localization is that fibrant simplicial presheaves are those that satisfy homotopy hyperdescent for the topology.

7 Homotopy Limits

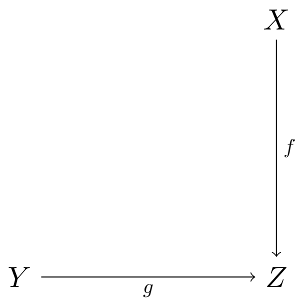
Example 4. To every morphism of spaces $f : X \to Y$ we can take the **homotopy fiber** of f, which is obtained by factoring f into an acyclic cofibration g and a fibration $\widetilde{f}$, such that $f : X \xrightarrow{\sim} \widetilde{X} \xrightarrow{\widetilde{f}} Y$, and then taking the fiber F of the fibration $\widetilde{f}$. From the fibration we get a long fibration sequence and this gives a long exact homotopy sequence

$$\cdots \to \pi_n(F) \to \pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(F) \to \cdots ,$$

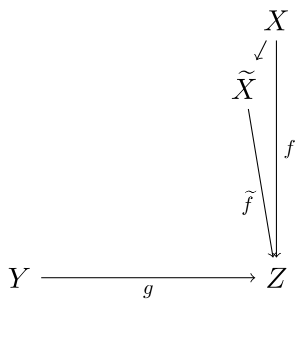
which is quite useful.

The homotopy fiber is a special kind of homotopy pullback, which is again a special kind of homotopy limit. A homotopy limit is a variant of the usual limit of a diagram, which is invariant under weak equivalences. We now give a hands-on definition of homotopy pullbacks, instead of discussing general homotopy limits.

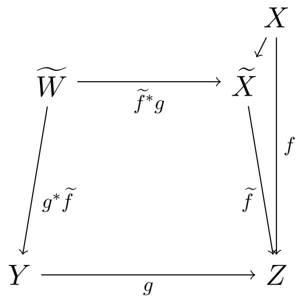
Definition 20. Given a diagram



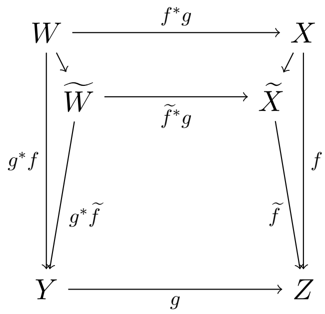
we could form a pullback, but first we replace the arrow f by a fibration $\widetilde{f}$:



and then we take a pullback of g along the fibration $\widetilde{f}$:



The resulting space $\widetilde{W}$, together with the morphisms to Y and $\widetilde{X}$ is called a **homotopy pullback** of the first diagram. If one forms the classical pullback of the diagram,



one calls the outer square a **homotopy pullback diagram** if the induced morphism $W \to \widetilde{W}$ is a weak equivalence.

This satisfies the axioms of a model category thanks to a theorem of Kan which also tells us that this category inherits the properties of being simplicial or/and proper from $\mathcal{M}$.

General homotopy (co)limits may be defined as derived functors:

Definition 21. Let I be a Reedy category and $\mathcal{M}$ a model category. Then the diagram category (or functor category) $\mathcal{M}^I$ can be equipped with the Reedy model structure. Denote by Q the cofibrant replacement and by R the fibrant replacement in this Reedy model category. Then for any diagram $\mathcal{F} \in \mathcal{M}^I$ we define

$$hocolim\mathcal{F} := colimQ\mathcal{F}, \qquad holim\mathcal{F} := \lim R\mathcal{F}.$$

8 h-Principles

Definition 22. Let X, Y be two spaces, and $\mathcal{F}, \mathcal{G}$ two classes of morphisms $X \to Y$, such that $\mathcal{F} \subset \mathcal{G}$. Then X, Y satisfy a **weak h-principle** with respect to $\mathcal{F} \subset \mathcal{G}$, if every morphism in $\mathcal{G}$ admits a homotopy to a morphism in $\mathcal{F}$.

Let $\mathcal{F}, \mathcal{G}$ be sheaves on a category of spaces, with a morphism $\mathcal{F} \to \mathcal{G}$. Then this data is said to satisfy an **h-principle** over some set of spaces if the sections over this set are weak equivalences. The former weak h-principle is the global section π_0-version.

Example 5. Let M, N be smooth manifolds, with M compact and N without boundary, and either M without boundary or $\dim M < \dim N$. Then to every immersion $M \to N$ we can associate a bundle map of tangent bundles, which is injective on total spaces $TM \hookrightarrow TN$. Any such bundle map is called a **formal immersion**. The Smale-Hirsch theorem states that M, N satisfy an h-principle with respect to the inclusion of immersions into the set of all formal immersions. One might also state this as:

$$Imm(M, N) \hookrightarrow Imm^f(M, N) \text{ is a weak equivalence.}$$

The benefit of this theorem is, that formal immersions are much better from a homotopy-theoretic point of view, since the space of all formal immersions fibers over the space of smooth maps $M \to N$, with fiber over some $f : M \to N$ being the space of injective maps $TM \to f^*TN$ over M. The space of smooth maps is homotopy equivalent to the space of all continuous maps (due to integrating against a smoothing kernel), whose homotopy type might be more accessible (for example vanish).

Another example is Gromov's h-principle for partial differential relations:

Definition 23. A **partial differential relation** of order q on maps $N^n \to M^m$ is a subset X of jet bundle germs $J_0^q(\mathbb{R}^n, M)$ that is invariant under isomorphisms of germs $(\mathcal{C}^\infty)_0(\mathbb{R}^n, M)$.

From these data we can define the sheaf of formal solutions $Sol_X^f(-, M)$ over N as a subsheaf of the q-jet sheaf $J^q(-, M)$ over N which satisfies X at every stalk. We call the sheaf of solutions $Sol_X(-, M)$ the preimage of the formal solutions under the morphism $\mathcal{C}^\infty(-, M) \to J^q(-, M)$.

We say that N and M satisfy an h-principle for the PDR X if $Sol_X(-, M) \hookrightarrow Sol_X^f(-, M)$ is a weak equivalence.

In fact, since being an immersion means to satisfy a partial differential relation of the first order (injective differential), Gromov's h-principle is a generalization of Smale-Hirsch.

Definition 24. A complex manifold X has the **Oka-Grauert property** if $\mathcal{O}(S, X) \hookrightarrow \mathcal{C}(S, X)$ is a weak equivalence for all Stein manifolds S.

8.1 Homotopy Sheaves and h-principles

A very rough idea to prove an h-principle is the following: Try to prove it locally, i.e. look at sections of the sheaves over small convenient sets. Then put the local solutions together somehow.

The classical formalism is to prove that the restrictions maps for certain sections are fibrations, when one regards the sheafs as sheafs of topological spaces (i.e. the sections are topological spaces). This leads to the notion of flexible sheaves. Alternatively, one can prove that the sheaves involved are homotopy sheaves.

Definition 25. Let $\mathcal{C}$ be a site and $\mathcal{F} : \mathcal{C} \to \Delta^{op}Set$ a simplicial presheaf. Then $\mathcal{F}$ is called a **homotopy sheaf** if for all coverings $\{U_i \to U\}$ of objects $U \in \mathcal{C}$,

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is a homotopy equalizer diagram.

Theorem 4 (Lárusson). A complex manifold X has the Oka-Grauert property iff it has finite homotopy excision, when considered as a representable simplicial presheaf in simplicial degree 0.