Cobordism spectra

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Overview

1. I will introduce the philosophy of stable homotopy theory, define a category of spectra, define homology and cohomology with coefficients in a spectrum. Here we'll pay some attention to the difference between spectra and Ω-spectra.
2. We define a cobordism spectrum (= a Thom spectrum) for unoriented real vector bundles MO. There it's important to know how Thom spaces behave under adding a trivial bundle.
3. I will review the proof of the theorem $\Omega_n \xrightarrow{\sim} \pi_n MO$.
4. We introduce the notion of singular manifolds inside another manifold and bordism of singular manifolds, which yields the group $\Omega_n(N)$ of singular manifolds in N, up to bordism in N.
5. I will prove that $\Omega_n(N) \xrightarrow{\sim} H^n(N, MO)$ - the cohomology of N with coefficients in the cobordism spectrum.

Overview

If time is left...

If there is some time left, I can explain how this generalizes to singular manifolds with X-structure, which gives a much more powerful theorem that applies to framed, oriented, complex, spin, string, whatever bordism classes as well. There are two things to do before one can generalize the proof: First, $\Omega_n(N)$ is juiced up to $\Omega_n^X(N)$; second, MO is juiced up to MX. The proof then is essentially a technical issue of using convenient notation.

Stable Homotopy Theory

Philosophy

Given a map $f : X \to Y$ one can look at the suspensions $\Sigma^k f : \Sigma^k X \to \Sigma^k Y$. If the map f was nullhomotopic, the $\Sigma^k f$ are nullhomotopic, too. However, for $\Sigma^k f$ being nullhomotopic, f needn't be nullhomotopic. The object $\Sigma^\infty f : \Sigma^\infty X \to \Sigma^\infty Y$ thus carries different (strictly less) information than the object f. For two finite CW complexes X, Y the set of homotopy classes $[\Sigma^k X \Sigma^k Y]$ eventually coincides with $[\Sigma^{k+1} X, \Sigma^{k+1} Y]$. This process is called *stabilization*. One can observe that homotopy groups of a space depend on the unstable information, i.e. $\pi_n(X) = [S^n, X] \neq [\Sigma^k S^n, \Sigma^k X]$ in general. On the other hand, (co)homology depends only on the stable information, i.e. $H^n(X; \mathbb{Z}) = H^{n+k}(\Sigma^k X; \mathbb{Z})$. Therefore, it can be helpful to work in a category where only the stable information matters, to get some knowledge about (co)homology of a space.

Definition

- A *spectrum* $\mathbb{E}$ is a sequence of spaces E_n together with continuous maps $\Sigma E_n \to E_{n+1}$.
- An Ω-*spectrum* is a spectrum $\mathbb{E}$ where $\Sigma E_n \to E_{n+1}$ corresponds to a weak homotopy equivalence $E_n \xrightarrow{\sim} \Omega E_{n+1}$ under the adjunction of Σ with Ω.
- Given a space X we define its *suspension spectrum* $\Sigma^\infty X$ to be the sequence $\Sigma^n X$ together with the identity maps $\Sigma\Sigma^n X \to \Sigma^{n+1} X$.
- To each spectrum $\mathbb{E}$ we associate its *fibrant replacement* which consists of the spaces $\lim_{k\to\infty} \Omega^k E_{n+k}$ together with the obvious maps.

Proposition

The fibrant replacement of any spectrum is an Ω-spectrum.

Proposition

For any Ω-spectrum $\mathbb{E}$ one can define (co)homology theories

$$H^n(X;\mathbb{E}) := [S^n \wedge X, E_0]$$

$$H_n(X;\mathbb{E}) := [S^n, X \wedge E_0]$$

that satisfy the Eilenberg-Steenrod axioms.

Theorem (Brown representability)

Every generalized Eilenberg-Steenrod (co)homology theory comes from an Ω-spectrum (= is representable in the category of spectra).

Definition

Given a spectrum $\mathbb{E}$, the n-th homotopy group of $\mathbb{E}$ is defined to be the n-th homotopy group of the 0-th space of the fibrant replacement, i.e.

$$\pi_n(\mathbb{E}) = [S^n, \lim_{k\to\infty} \Omega_k E_k] = [S^{n+k}, E_k] = \pi_{n+k}(E_k) \text{ for } k \gg 0$$

Remark

The homology and cohomology of a point, with coefficients in (the fibrant replacement of) $\mathbb{E}$, are the same as the homotopy groups of $\mathbb{E}$.

The Thom Spectrum
(for unoriented real vector bundles) – I

The inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \oplus \mathbb{R}^1$ yields an inclusion $O(n) \hookrightarrow O(n+1)$ (by acting trivial on the extra $\mathbb{R}^1$ factor). This yields a map of classifying spaces $BO(n) \to BO(n+1)$ which can be concretely seen in the model $BO(n) = Gr(n, \infty)$ of the Grassmannian of n-planes in $\mathbb{R}^\infty$ as the map $g_n : Gr(n, \infty) \to Gr(n+1, \infty)$ that maps a subspace $V \subset \mathbb{R}^N$ to the subspace $V \oplus \mathbb{R}^1 \subset \mathbb{R}^N \oplus \mathbb{R}^1$. Denote by $\gamma^n : \mathbb{R}^\infty \to Gr(n, \infty)$ the tautological (universal) $O(n)$-bundle. The map g_n induces a bundle map $g_n : g_n^* \gamma^{n+1} \to \gamma^n$.

Proposition

$g_n^* \gamma^{n+1} \simeq \gamma^n \oplus \epsilon^1$.

The Thom Spectrum
(for unoriented real vector bundles) – II

Proposition

$Th(g_n^*\gamma^{n+1}) \simeq Th(\gamma^n \oplus \epsilon^1) \simeq \Sigma Th(\gamma^n)$.

Proposition

The maps g_n induce continuous maps of Thom spaces

$$Th(g_n) : Th(g_n^*\gamma^{n+1}) \to Th(\gamma^{n+1})$$

Definition

Denote $MO_n := Th(\gamma^n)$ then by MO we denote the spectrum formed by the spaces MO_n together with the maps $Th(g_n) : \Sigma MO_n \to MO_{n+1}$, called *Thom spectrum* of O, or just *unoriented cobordism spectrum*, sometimes $\mathcal{N} := MO$.

Thom's Theorem

Absolute Case (over a point)

Theorem (Thom)

$$\Omega_n \xrightarrow{\sim} H_n(pt; MO) = [S^{n+k}, MO_k] \qquad \text{(for } k \gg 0\text{).}$$

We will generalize this statement by replacing the point pt by an arbitrary space N, which requires us to replace the left hand side of the isomorphism as well, by something called $\Omega_n(N)$, to be defined later.

We have already seen the proof; However, we revise the main steps of the proof now.

Thom's Theorem

Absolute Case – Proof Structure

a) To a manifold M associate $f_M := D(\nu) \subset S^{n+k} \to Th(\gamma^k) = MO_k,\ S^{n+k} \setminus D(\nu) \to * \subset MO_k$ for an embedding $M \hookrightarrow \mathbb{R}^n$ with normal bundle ν.

b) See that the homotopy class of f_M depends only on the diffeomorphism class of M, get a map $\phi : \{n\text{-manifolds}\}/_\sim \to \pi_n MO$.

c) Disjoint union is mapped to wedge sum (pinching trick).

d) Bordisms are mapped to homotopies under ϕ, since nullbordant manifolds are mapped to 0; we get a group homomorphism $\Phi : \{\text{compact } n\text{-manifolds}\}/_{\text{cob}} \to \pi_n MO$.

e) Surjectivity via the transversality trick: pick $f \in \alpha \in \pi_n MO$ such that $M := f^{-1}(BO(k))$ does the job, considering $BO(k) \hookrightarrow MO_k$ as zero section.

f) Injectivity via the transversality trick: homotopy h_t yields $h^{-1}(BO(k))$, bordism from $h_0^{-1}(BO(k))$ to $h_1^{-1}(BO(k))$.

Let N be a fixed topological space.

Definition

- A *singular n-manifold in N* is a continuous map $s : M \to N$ with M an n-manifold. It is called *compact*, if M is compact.
- Two singular n-manifolds $s : M \to N$, $s' : M' \to N$ are said to be diffeomorphic, if there exists a diffeomorphism $t : M \xrightarrow{\sim} M'$ such that $s' \circ t = s$.
- A *bordism of compact singular n-manifolds* $s : M \to N$, $s' : M' \to N$ is a singular $(n+1)$-manifold $w : W \to N$ with boundary $\partial W = M \sqcup M'$ such that $w|M = s$ and $w|M' = s'$.
- We denote $\Omega_n(N)$ the bordism classes of compact singular n-manifolds in N.

Some observations:

- The map $N \mapsto \Omega_n(N)$ is contravariant functorial, i.e. to every continuous map $n : N \to N'$ we can associate a group homomorphism $\Omega_n(N') \to \Omega_n(N)$ by composing with n.
- $\Omega_n(\cdot)$ maps wedge sums to direct sums: $\Omega_n(\bigvee N_\alpha) = \bigoplus \Omega_n(N_\alpha)$
- $\Omega_n(pt) \neq 0$ for all n.

Conclusion: $\Omega_n(\cdot)$ behaves a lot like an Eilenberg-Steenrod cohomology theory, but with coefficients different from singular cohomology.

In fact, Poincaré's first attempt to define cohomology looked very much like $\Omega_n(\cdot)$.

The Relative Thom's Theorem

Theorem (Thom)

$$\Omega_n(N) \xrightarrow{\sim} H_n(N; MO) = [S^{n+k}, N \wedge MO_k] \qquad \textit{(for } k \gg 0\textit{).}$$

Proof:

We discuss only the deviation from the absolute case.

a) To $s : M \to N$ associate $f_s := f_M \wedge (s \circ p) : S^{n+k} \to MO_k \wedge N$, where p is the projection $D(\nu) \to M$ and $*$ on $S^{n+k} \setminus D(\nu)$.

b) If $s : M \to N$, $s' : M' \to N$ are diffeomorphic via $t : M \to M'$, their stable normal bundles can be represented by the same space, hence $f_M = f_{M'}$. Furthermore, $t \circ p = p'$ and $s = s' \circ t$ imply that $s \circ p = s' \circ t \circ p = s' \circ p'$.

The Relative Thom's Theorem

Proof continued:

c) $\sqcup \mapsto \vee$ with the same argument (pinching trick).

d) If $w : W \to N$ is a bordism of $s : M \to N$ with the trivial singular manifold $s' : \emptyset \to N$, we have f_M nullhomotopic, hence $f_M \wedge (s \circ p)$ is nullhomotopic as well. This establishes a group homomorphism

$$\Phi : \Omega_n(N) \to \pi_{n+k}(MO_k \wedge N), \qquad (s : M \to N) \mapsto [f_s]$$

The Relative Thom's Theorem

Proof continued:

e) Surjectivity (with the transversality trick): choose a representative $f \in [f] \in \pi_{n+k}(MO_k \wedge N)$ which is smooth and transversal to the zero section $Gr(k, N) \subset Gr(k, \infty) = BO(k) \hookrightarrow MO_k$ wedged with the identity on N, then $M := f^{-1}(Gr(k, N) \wedge N)$ is a smooth manifold of codimension k in S^{k+n}, i.e. an n-dimensional manifold. It has a natural map $s : M \to N$, which is f followed by the projection to N. One can check that this s has f_s equal to f, if one chooses the embedding $M \to S^{n+k}$ to represent the stable normal bundle.

f) Injectivity is similar: If $s : M \to N$ and $s' : M' \to N$ have f_s and $f_{s'}$ homotopic via homotopy $h : S^{n+k} \times I \to MO_k \wedge N$, one can choose a smooth map homotopic to h which is transversal enough to make $W := h^{-1}(Gr(k, N) \wedge N)$ a smooth submanifold of $S^{n+k} \times I$. This W is a bordism from s to s'. $\square$

Now that we know that $\Omega_\bullet(\cdot)$ is just MO-homology, what can we do with it?

A homology theory admits a long exact sequence for pairs $A \hookrightarrow N$:

$$\cdots \to \Omega_n(A) \to \Omega_n(N) \to \Omega_n(N/A) \to \Omega_{n-1}(A) \to \Omega_{n-1}(N) \to \cdots$$

We could use homology operations to study bordism groups.
We can see if MO-cohomology also has a geometric meaning.
We can do the same for oriented bundles, then $\Omega^{SO}_\bullet \simeq H_\bullet(\cdot; MSO)$.

- Let X be a sequence of spaces X_n with maps $X_n \to X_{n+1}$ (unlike a spectrum!) and fibrations $F_n : X_n \to BO(n) = Gr(n, \infty)$ that commute with $BO(n) \to BO(n+1)$. The pullback $F_n^* \gamma_n$ is called the X-universal bundle, its Thom space MX_n organises into a spectrum MX, the *Thom spectrum for* X, or just X-cobordism spectrum.
- An X-manifold is a triple $(M, h, \tilde{\nu})$ with $h : M \hookrightarrow \mathbb{R}^{n+k}$ an embedding, $\nu : M \to BO_n$ classifying the stable normal bundle and $\tilde{\nu} : M \to X_n$ a chosen F_n-lift of it, i.e. $F_n \circ \tilde{\nu} = \nu$.
- An X-map $(M, h, \tilde{\nu}) \to (M', h', \tilde{\nu'})$ is a smooth map $g : M \to M'$ such that there is a translation $T : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ with $h' \circ f = T \circ h$ and there exists a homotopy of $\tilde{\nu'} \circ f$ with $\tilde{\nu}$ that lifts $\nu' \circ f = \nu$.

Oriented bordism

For $X_n = BSO(n) = \tilde{Gr}(n, \infty)$, the oriented Grassmannian, there is a natural $X_n \to BO(n)$, which is the twofold cover $\tilde{Gr}(n, \infty) \to Gr(n, \infty)$. The datum of an X-structure on a manifold coincides with an orientation on the stable normal bundle.

Framed bordism

For $X_n = pt$, the one-point space, there is the map $X_n \to BO(n)$ sending everything to the basepoint. The datum of an X-structure on a manifold means that the map classifying the stable normal bundle factors over the one-point space, so it is trivial. Hence, admitting an X-structure means having a stably trivial normal bundle, i.e. being parallelizable.

X-Structures

on singular manifolds

- A singular X-manifold in N is just a singular manifold $s : M \to N$ with an X-structure on M.
- A map of singular X-manifolds in N is just a map of X-manifolds that commutes with the maps to N.
- An X-bordism of singular X-manifolds M, M' is a singular X-manifold of higher dimension with boundary $M \sqcup -M'$, with induced X-structures those of M resp. $-M'$, where the sign denotes orientation reversal of the embedding map $h : M' \hookrightarrow \mathbb{R}^{n+k}$.
- We denote $\Omega_n^X(N)$ the group of singular X-manifolds in N up to X-bordism.

Thom's Theorem for X-Structures

Theorem (Thom)

$$\Omega_n^X(N) \xrightarrow{\sim} H_n(N; MX)$$

Corollary (Pontryagin-Thom)

$$\Omega_n^{fr}(pt) \xrightarrow{\sim} H_n(pt; M(pt)) = \pi_n^{stab}$$

Proof.

Follow the same steps as before, occasionally composing with the map F_n induces on Thom spaces $MX_k \to MO_k$. The only crucial ingredient to remember is that F_n was a fibration, hence one can apply lifting criteria. □

Cobordism Cohomology

To study $H^n(N; MX) = [S^{n+k} \wedge N, MX]$ one can first try to get something geometric by applying the "surjectivity" part of the previous construction. For simplicity, now $X = BO$, hence $MX = MO$. Take $f : S^{n+k} \wedge N \to MO_k$, compose with the projection $S^{n+k} \times N \to S^{n+k} \wedge N$, homotope to get something transversal enough to make $M := f^{-1}(BO_k) \subset S^{n+k} \times N$ a smooth codimension k manifold. It comes with a map $s : M \to N$, the restriction of the projection to N. To get an X-structure on M, we would repeat this process with a tubular neighbourhood of BO_k. In the end, we get an isomorphism

$$\Omega^X_{n-\dim N}(N) \xrightarrow{\sim} H^n(N; MX)$$

which looks similar to Poincaré duality.