Model Categories: Modern Algebraic Topology

Konrad Voelkel*

Model categories are the most prominent formalism for modern algebraic topology, let it be the study of topological spaces, simplicial sets or homological algebra. It allows a precise understanding of the similarity between the mapping cone in homotopy theory and the mapping cone in triangulated categories, to give just one arbitrary example. One also encounters model cats in mathematical physics (n-categories), algebraic geometry (derived ag, motivic homotopy theory), differential topology (h-principles), K-theory, (modular) representation theory ...

We want to have a short look at the basic terms of this language and the elementary principles that justify using another layer of abstraction.

1 The abstract situation

Let $\mathcal{C}$ be a category which has all finite limits and colimits (that includes products and direct sums, in particular the existence of an initial and a terminal object). For example, $\mathcal{C}$ could be the category of topological spaces or the category of chain complexes over an abelian category.

Let W be a class of morphisms in $\mathcal{C}$, for example homotopy equivalences of topological spaces, or quasi-isomorphisms of chain complexes.

Then $\mathcal{HC} := [W^{-1}]\mathcal{C}$, the **homotopy category**, is a category one wants to study. In the case of chain complexes, the homotopy category is the elusive derived category. Usually it is quite hard to understand such a localization, as it is a very abstract thing. We like to work with something more concrete, something more computable!

Very roughly,

$$\text{model cat} \;::\; \text{homotopy cat} \qquad \approx \qquad \text{basis} \;::\; \text{vector space.}$$

The name "model cat" comes from the fact that there can be many model structures to study the same localization, just as there are many bases for the same vector space.

2 Resolutions of objects

A useful technique in homological algebra are injective or projective resolutions. In classical homotopy theory one can use cellular approximation, which assigns to each space a CW complex with the same homotopy groups. These concepts have a common, quite powerful description: **fibrant** and **cofibrant resolutions**.

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The axioms of a model category imply that one can assign functorially to each object such a resolution, which is a morphism into or from another object with better properties.

Since Grothendieck, we're convinced that one should work in a relative setup rather than absolutely, which requires that one doesn't look just at objects X as objects themselves but as morphisms from the initial object into it: $\emptyset \to X$ or from the object to the terminal object: $X \to \mathsf{pt}$. Then we can easily generalize to $Y \to X$ or $X \to Y$ for any Y, which gives us more freedom.

The good properties of a fibrant or cofibrant object X are expressed in terms of properties of the maps $\emptyset \to X$ and $X \to \mathsf{pt}$. These properties of maps are also defined for arbitrary maps $Y \to X$ and $X \to Y$.

Altogether, this leads to the notion of a **fibration** or **cofibration**. A model structure on a category $\mathcal{C}$ with a class of weak equivalences W is nothing but the choice of a designated class of morphisms called fibrations and a designated class of morphisms called cofibrations, such that some axioms are satisfied.

3 Hand-waving the axioms

Let $\mathcal{M}$ be a category with three classes of morphisms W, C, F. Then we call this a model category if the following axioms are satisfied:

- M1 The cat is complete and cocomplete (has all finite limits and colimits).
- M2 2-out-of-3 for weak equivalences.
- M3 Retracts of W,C,F are again W,C,F respectively.
- M4 The Lifting-Extension-Property holds.
- M5 There are two functorial factorizations of morphisms, into a cofibration followed by a fibration. In each, one of the two is also a weak equivalence.

Instead of making this more precise, we recommend Wikipedia and look at some examples.

4 Example: topological spaces

Suppose you have a relative CW complex, that is a pair of spaces $A \subset X$ such that X is obtained from A by gluing in disks along their boundary iteratively. Then it is always interesting to ask whether a given map $A \to Y$ extends to a map $X \to Y$.

Suppose you have a covering map $\widetilde{Y} \to Y$, then it is always interesting to ask whether a given map $X \to Y$ can be lifted to a map $X \to \widetilde{Y}$.

Combining these, you get the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{Y} \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & Y \end{array}$$

The lifting-extension axiom of model categories (which holds in this situation) says: if either the left or the right column is a weak homotopy equivalence, then there is such a lift.

5 Example: chain complexes

Let $\mathcal{M} := Ch_{\geq 0}(\mathcal{A})$ be the category of non-negatively graded chain complexes over an abelian category $\mathcal{A}$ with quasi-isomorphisms (i.e. morphisms that become isos on homology) as weak equivalences. For example, $\mathcal{A} = R\text{-}Mod$.

We define the fibrations to be morphisms which are in each positive degree $\mathcal{A}$-epimorphisms, and cofibrations to be morphisms which are in each degree $\mathcal{A}$-monos with $\mathcal{A}$-projective cokernels. It is a theorem that this forms a model category.

The functorial factorization axiom of model categories tells us that for each morphism $C_\bullet \to D_\bullet$ in $\mathcal{M}$ there is a factorization $C_\bullet \xrightarrow{cof} \widetilde{D}_\bullet \xrightarrow{fib} D_\bullet$ such that the fibration is also a weak equivalence.

Now look at a morphism $0 \to D_\bullet$ and apply this, then we get a quasi-isomorphism $\widetilde{D}_\bullet \to D_\bullet$ with the properties that $0 \to \widetilde{D}_\bullet$ has a degreewise projective cokernel, i.e. $\widetilde{D}_\bullet$ consists of projective objects, and $\widetilde{D}_\bullet \to D_\bullet$ is degreewise an epimorphism, except possibly in degree 0. This is precisely a projective resolution of D.

Another comment about a model category of chain complexes: the mapping cone is now required to be a functor on morphisms, which is not the case for triangulated categories in general. In this sense, the formalism is better to work with.

6 What are these good for?

A naive notion of "homotopy" is either left homotopy defined in terms of cylinder objects, or right homotopy defined in terms of path objects. These don't even give equivalence relations in general, and if, then they don't coincide. For cofibrant-fibrant objects, both notions coincide and define an equivalence relation!

The main point of model categories to understand localizations is: if we want to understand $[X, Y]$, the morphisms in the localized category, we can take cofibrant-fibrant resolutions $\widetilde{X}$ and $\widetilde{Y}$ and look at honest maps $\widetilde{X} \to \widetilde{Y}$ up to the homotopy equivalence relation.