Serre's Problem on Projective Modules

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6. February 2013

The main source for this talk was Lam's book "Serre's problem on projective modules". It was Matthias Wendt's idea to take the cuspidal cubic curve to construct a counterexample to homotopy invariance of vector bundles on a singular affine variety. The graphics and diagrams are drawn using TikZ/PGF.

1 Setting

Notation

Fix a ring R and a field k (you can take $R = k = \mathbb{C}$ if you'd like to).

"vector bundle" will mean "algebraic k-vector bundle".

For a variety X/R denote by $VB(X)$ the set of isomorphism classes of vector bundles $\mathcal{E}$ on X. This is a contravariant functor by mapping a morphism of varieties $f : X \to Y$ to the pullback map $f^* : VB(Y) \to VB(X)$.

For any variety X/R denote by $X \times_R \mathbb{A}^1{}_R := X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R[t])$ the affine line over X. It comes with a canonical projection morphism $pr_X : X \times_R \mathbb{A}^1{}_R \to X$.

We say that a vector bundle $\mathcal{E}$ over $X \times_R \mathbb{A}^1{}_R$ is **extended** from X if there is a vector bundle $\mathcal{F}$ over X such that $(pr_X)^*\mathcal{F} \simeq \mathcal{E}$, i.e. if the isomorphism class $[\mathcal{E}] \in VB(X \times_R \mathbb{A}^1{}_R)$ is in the image of $(pr_X)^*$.

We say that the functor $VB(-)$ is **homotopy invariant** on a subcategory $\mathcal{C}$ of all varieties if for all $X \in \mathcal{C}$ the projection pr_X induces a bijection $(pr_X)^* : VB(X \times_R \mathbb{A}^1{}_R) \xrightarrow{\sim} VB(X)$, i.e. if all vector bundles over the affine line over X are extended.

2 Some questions on homotopy invariance

Q 1. Is $VB(-)$ homotopy invariant on all quasiprojective varieties?

A. No. Homotopy invariance fails for the smooth projective variety $\mathbb{P}^1$.

Proof. We construct for each $a \in \mathbb{Z}$ a rank 2 vector bundle $\mathcal{E}(a)$ on $\mathbb{P}^1 \times \mathbb{A}^1$ which is not extended, by gluing two trivial vector bundles on $\mathbb{A}^1 \times \mathbb{A}^1$ via an explicit transition function $\left(\mathbb{A}^1 \setminus \{0\}\right) \times \mathbb{A}^1 \to \mathrm{GL}_2,\ (z,t) \mapsto A_{z,t}$ given by

$$A_{z,t} := \begin{pmatrix} z^a & tz \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k[z, z^{-1}, t]).$$

Claim.

$$\mathcal{E}(a)|_{\mathbb{P}^1 \times 0} \overset{(1)}{\simeq} \mathcal{O}(-a) \oplus \mathcal{O} \overset{(3)}{\not\simeq} \mathcal{O}(-(a-1)) \oplus \mathcal{O}(-1) \overset{(2)}{\simeq} \mathcal{E}(a)|_{\mathbb{P}^1 \times 1}.$$

Proof of Claim.

1. $A_{z,0} = z^a \oplus 1$ defines $\mathcal{O}(-a) \oplus \mathcal{O}$.
2. $A_{z,1} = \begin{pmatrix} z^a & z \\ 0 & 1 \end{pmatrix}$ and in another trivialization

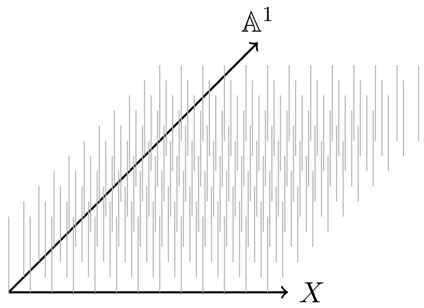
   $$\begin{pmatrix} z^{-1} & -1 \\ 1 & 0 \end{pmatrix} A_{z,1} \begin{pmatrix} 1 & 0 \\ -z^{a-1} & 1 \end{pmatrix} = \begin{pmatrix} z^{a-1} & 0 \\ 0 & z \end{pmatrix}$$

   which defines $\mathcal{O}(-(a-1)) \oplus \mathcal{O}(-1)$.
3. By a theorem of Grothendieck (or using older, less popular theorems), vector bundles on $\mathbb{P}^1$ always decompose uniquely (up to permutation) into a sum of line bundles and two vector bundles are isomorphic iff the line bundles in their decomposition are isomorphic (up to permutation).

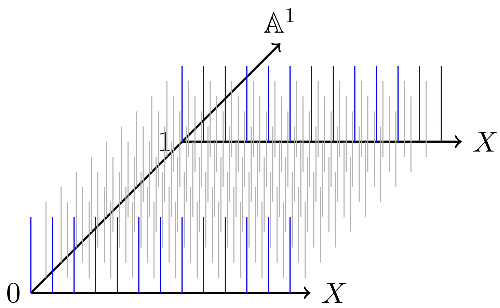
□

Remark. Topologically, there is a homotopy from the matrix $A_{z,t}$ to $A_{z,0}$, which corresponds to a homotopy of the classifying map $\mathbb{P}^1\mathbb{C} \times \mathbb{C} \to Gr_\infty$ of the bundle $\mathcal{E}(a)$ to a map which is constant along the $\mathbb{C}$ factor. Analogously we can show that, as topological vector bundles, $\mathcal{O}(k) \oplus \mathcal{O}(l) \simeq \mathcal{O}(k+l) \oplus \mathcal{O}$, but not algebraically.

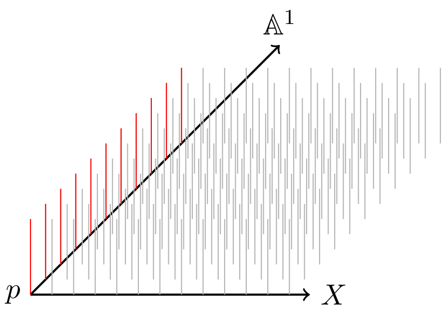
Illustration of families of vector bundles



A bundle over $X \times \mathbb{A}^1$.



If the restrictions to $X \times \{0\}$ and $X \times \{1\}$ are non-isomorphic, the whole bundle is not extended from X.



Removing a point $p \in X$, one can restrict the bundle to $\mathbb{A}^1 \times X \setminus \{p\}$ and ask whether this bundle is extended from X.

Q 2. Is $VB(-)$ homotopy invariant on all affine varieties?

A. No. Homotopy invariance fails for the singular affine curve $C^{aff} := \{y^2 - x^3 = 0\} \subset \mathbb{A}^2$ "cuspidal cubic".

Proof. We work with the projective curve $C := \{zy^2 - x^3 = 0\} \subset \mathbb{P}^2$, where $0 = [0 : 0 : 1] \in C$ is the singular point and $\infty = [0 : 1 : 0] \in C$ the point at infinity, such that $C^{aff} = C \setminus \{\infty\}$. The nonsingular part $C_{ns} = C \setminus \{0\}$ has a group structure (constructed like for an elliptic curve), and is isomorphic (as algebraic group) to $\mathbb{G}_\mathrm{a}$; fix an isomorphism $\varphi : \mathbb{G}_\mathrm{a} \xrightarrow{\sim} C_{ns}$. Furthermore (see [Hartshorne, Example II.6.11.4]), there are group isomorphisms $C_{ns} \xrightarrow{\sim} CaCl^\circ(C) \xrightarrow{\sim} Pic^\circ(C),\ p \mapsto \mathcal{O}(\infty - p)$.

Now we take the graph $\Gamma_\varphi \subset \mathbb{A}^1 \times C$ of $\varphi : \mathbb{A}^1 \hookrightarrow C$, this is a divisor, so we can take the line bundle $\mathcal{O}(\Gamma_\varphi)$ on $\mathbb{A}^1 \times C$. If we pull back along $t : \{t\} \times C \hookrightarrow \mathbb{A}^1 \times C$, we get

$$t^*\mathcal{O}(\Gamma_\varphi) \simeq \mathcal{O}(\infty - \varphi(t))$$

as one can see from the local equation. Since the $\mathcal{O}(\infty - \varphi(t))$ are non-isomorphic for different t, this shows that $\mathcal{O}(\Gamma_\varphi)$ is not extended from C.

If we restrict $\mathcal{O}(\Gamma_\varphi)$ to $\mathbb{A}^1 \times C^{aff}$, the fibers over $t \in \mathbb{A}^1$ are still non-isomorphic (since $Pic(C^{aff}) \simeq \mathbb{A}^1$, as one can see using a Mayer-Vietoris argument for K_0 on the normalization of the curve), hence we have an explicit bundle on $\mathbb{A}^1 \times C^{aff}$ that is not extended from C^{aff}. $\square$

Q 3. (Serre's Problem on Projective Modules) Are all finitely generated projective modules over a polynomial ring $k[t_1, \dots, t_n]$ free?

A. Yes, this is the Quillen-Suslin theorem.

Some questions we're not going to answer in detail here

Q 4. Is $VB(-)$ homotopy invariant on all smooth affine varieties?

A. Yes, that's a theorem by Lindel and others. The proof idea is more or less that a smooth affine variety looks étale-locally like $\mathbb{A}^n_k$, where one can use Quillen-Suslin (but it's not *that* easy).

Q 5. For G a linear algebraic group, is every G-bundle on $\mathbb{A}^n_k$ trivial?

A. No, this depends heavily on G and there are not many positive results aside from Quillen-Suslin for $G = GL_n$.

Q 6. For R a regular local ring, are all finitely generated projective modules over a polynomial ring $R[t_1, \dots, t_n]$ extended from R?

This is the Bass-Quillen conjecture, it is still open.

The Picard group of the affine cuspidal cubic curve

In this section, we sketch how to prove that $Pic(C^{aff}) \simeq k$, which was used in the previous section to get a counterexample to homotopy invariance, not only from the cuspidal cubic in the singular projective case, but also in the singular affine case.

The ring of functions on the affine cuspidal cubic curve is $A := k[x,y]/(y^2 - x^3)$. We write $A = k[t^2, t^3]$ and the normalization is just $k[t^2, t^3] \hookrightarrow k[t] =: \widetilde{A}$.

The conductor (by definition, the annihilator of $\widetilde{A}/A$ as A-module) is $\mathfrak{c} = (t^2, t^3)$. The conductor square

$$\begin{array}{ccc} A & \hookrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ A/\mathfrak{c} & \hookrightarrow & \widetilde{A}/\mathfrak{c} \end{array}$$

is a pullback square and there is a Mayer-Vietoris exact sequence

$$0 \to A^\times \to \widetilde{A}^\times \oplus (A/\mathfrak{c})^\times \to \left(\widetilde{A}/\mathfrak{c}\right)^\times \to Pic(A) \to Pic(\widetilde{A}) \oplus Pic(A/\mathfrak{c})$$

where we know $A^\times = k^\times$, $\widetilde{A}^\times = k[t]^\times = k$, $A/\mathfrak{c} = k$ and one can show $\left(\widetilde{A}/\mathfrak{c}\right)^\times = \left(k[t]/(t^2, t^3)\right)^\times \simeq k^\times \oplus k$. Furthermore, $Pic(k) = 0$ and $Pic(k[t]) = 0$. Therefore, the interesting part of the exact sequence is

$$k^\times \oplus k^\times \to k^\times \oplus k \to Pic(A) \to 0$$

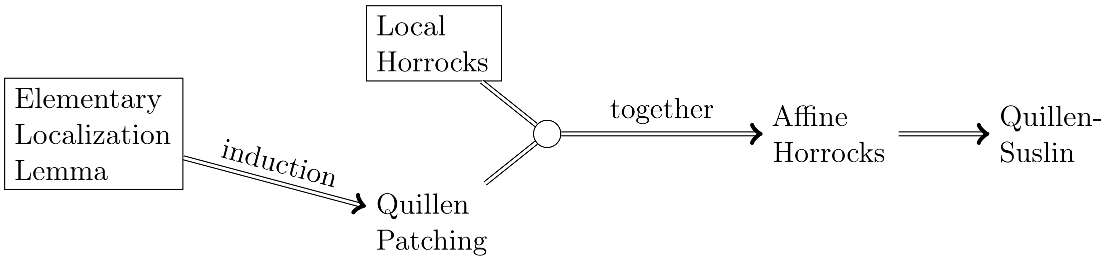
and analysis of the map shows that $k^\times \oplus k^\times$ maps precisely onto the $k^\times$ factor, so $k \xrightarrow{\sim} Pic(A)$. □

The statements we didn't prove so far can be shown "by hand", see for example Victor I. Piercey: "Picard Groups of Affine Curves".

3 Overview of Quillen's Proof

Theorem (Quillen-Suslin '76). Let R be a PID, $n \in \mathbb{N}$. Then any finitely generated projective module over $R[t_1, \dots, t_n]$ is free.

In other words: all vector bundles over $\mathbb{A}^n_R$ are trivial.



The proof falls out of an **affine Horrocks theorem** which is proved via **Quillen Patching** applied to a **local Horrocks theorem**. Quillen Patching is proved by a nested induction, using an **elementary lemma on localization** to start the induction. We will use local Horrocks and the localization lemma as black boxes.

We need some notation first:

Definition. We use the notation $\mathcal{M}(R)$ for the set of all finitely generated modules over a ring R and $\mathcal{P}(R)$ for the projective modules therein. If A is an R-algebra and $M \in \mathcal{M}(R)$, then we say that $A \otimes_R M \in \mathcal{M}(A)$ is **extended** from M and R and we write $Q \in \mathcal{M}^R(A)$ for all $Q \in \mathcal{M}(A)$ which are extended from R, and $\mathcal{P}^R(A)$ likewise.

In the light of Serre-Swan, this notion of "being extended" is compatible with the previous definition for vector bundles.

Theorem (Quillen Patching). Let R be any commutative ring, A an R-algebra and $M \in \mathcal{M}(A[t_1, \dots, t_n])$ finitely presented. Then

(A_n) $\quad Q(M) := \{g \in R \mid M_g \in \mathcal{M}^{A_g}(A_g[t_1, \dots, t_n])\}$ is an ideal of R, and

(B_n) $\quad \big(\forall \mathfrak{m} \in \operatorname{Max}(R) : M_\mathfrak{m} \in M^{A_\mathfrak{m}}(A_\mathfrak{m}[t_1, \dots, t_n])\big) \implies M \in \mathcal{M}^A(A[t_1, \dots, t_n])$.

The set $Q(M)$ is called the **Quillen ideal** of M.

Corollary. $P \in \mathcal{P}(R[t_1, \dots, t_n)$ is extended from R iff $\forall \mathfrak{m} \in \operatorname{Max}(R) : P_\mathfrak{m}$ is a free $R_\mathfrak{m}[t_1, \dots, t_n]$-module.

Geometrically, this means: an algebraic vector bundle on $\mathbb{A}^n \times X$ is extended from $X = \operatorname{Spec}(R)$ iff this is the case for a neighborhood of each closed point of X.

Proof of corollary. We specialize the theorem to $A = R$ and finitely generated projective modules M.

"$\Leftarrow$": Free modules over $R_\mathfrak{m}[t_1, \dots, t_n]$ are clearly extended from $R_\mathfrak{m}$, so by (B_n) P is extended from R.

"$\Rightarrow$": $P_\mathfrak{m}$ is extended from $P_\mathfrak{m}/(t_1, \dots, t_n)P_\mathfrak{m} \simeq (P/(t_1, \dots, t_n)P)_\mathfrak{m}$, which is projective over a local ring, hence free, so $P_\mathfrak{m} = (P/(t_1, \dots, t_n)P)_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}[t_1, \dots, t_n]$ is free. $\square$

Notation Denote by $R\langle t\rangle := R[t]_S$ the localization of $R[t]$ at the multiplicative set S of all monic polynomials in t. Monic means leading coefficient 1. Write $M\langle t\rangle := M_S$ for an $R[t]$-module M.

Fact. If R is a PID, then $R\langle t\rangle$ is a PID.

Theorem (Local Horrocks). Let R be a commutative *local* ring and $P \in \mathcal{P}(R[t])$.
If $P\langle t\rangle := R\langle t\rangle \otimes_{R[t]} P$ is $R\langle t\rangle$-free, then P is $R[t]$-free.

Theorem (Affine Horrocks). Let R be any commutative ring and $P \in \mathcal{P}(R[t])$.
If $P\langle t\rangle = R\langle t\rangle \otimes_{R[t]} P \in \mathcal{P}^R(R\langle t\rangle)$, then $P \in \mathcal{P}^R(R[t])$.

Remark. The geometric meaning of the Horrocks' Theorems is the following: If a vector bundle over $\mathbb{A}^1{}_R$ extends to $\mathbb{P}^1_R$, then it is extended from $\operatorname{Spec}(R)$.

Proof of Affine Horrocks. Let $P \in \mathcal{P}(R[t])$ with $P\langle t\rangle \in \mathcal{P}^R(R\langle t\rangle)$. For $\mathfrak{m} \in \operatorname{Max}(R)$, $P\langle t\rangle_\mathfrak{m} \in \mathcal{P}^{R_\mathfrak{m}}(R_\mathfrak{m}\langle t\rangle)$ and that implies $P\langle t\rangle_\mathfrak{m}$ is $R_\mathfrak{m}\langle t\rangle$-free. By Local Horrocks for $R_\mathfrak{m}$, $P_\mathfrak{m}$ is $R_\mathfrak{m}[t]$-free. By Quillen Patching (B), P is extended from R. □

The following proof of Quillen-Suslin via Affine Horrocks is due to Murthy.

Proof of the Quillen-Suslin Theorem. We proceed by induction over n, the base $n = 0$ is trivial. Let $A := R[t_2, \dots, t_n]$ and consider $A[t_1] \subset R\langle t_1\rangle[t_2, \dots, t_n] \subset A\langle t_1\rangle$.

If $P \in \mathcal{P}(R[t_1, \dots, t_n])$, then $P \otimes_{R[t_1,\dots,t_n]} R\langle t_1\rangle[t_2, \dots, t_n]$ is a finitely generated $R\langle t_1\rangle[t_2, \dots, t_n]$-module, by the induction hypothesis a free one. Hence, $P \otimes_{A[t_1]} A\langle t_1\rangle$ is a free $A\langle t_1\rangle$-module. Affine Horrocks implies P is extended from $P/t_1P \in \mathcal{P}(A)$. Again by the induction hypothesis, P/t_1P is A-free, so that P is $A[t_1]$-free. □

4 Proof of Quillen Patching

Proof of Quillen Patching. The proof proceeds in three steps.

1. $(A_n \implies B_n)$,
2. $(A_1 \implies A_n)$ by induction,
3. (A_1) using a localization lemma.

Step 1: It suffices to check: Assume (A_n), then for M as in B_n, we have $Q(M) = (1)$.

Let $M' := A[t_1, \ldots, t_n] \otimes_A (M/(t_1, \ldots, t_n)M)$, this is a finitely presented $A[t_1, \ldots, t_n]$-module which is extended from A.

For any $\mathfrak{m} \trianglelefteq R$ maximal there is an iso $\varphi : M_\mathfrak{m} \xrightarrow{\sim} M'_\mathfrak{m}$, since $M_\mathfrak{m}$ extended from $A_\mathfrak{m}$ means

$$M_\mathfrak{m} \simeq A_\mathfrak{m}[t_1, \ldots, t_n] \otimes_{A_\mathfrak{m}} (M_\mathfrak{m}/(t_1, \ldots, t_n)M_\mathfrak{m}) .$$

Now φ is the localization of an $A_g[t_1, \ldots, t_n]$-isomorphism $M_g \xrightarrow{\sim} M'_g$ for some $g \in R \backslash \mathfrak{m}$ (this is a common-denominator-trick). So, $g \in Q(M) \setminus \mathfrak{m}$, hence $Q(M) \not\subset \mathfrak{m}$. Every proper ideal is contained in some maximal ideal, $Q(M)$ is an ideal by assumption, hence $Q(M)$ is not a proper ideal but (1).

This shows $(A_n \implies B_n)$.

Step 2: We prove (A_n) by induction, assuming (A_1). The induction hypothesis is (A_{n-1}), hence (B_{n-1}) (by step 1).

Let M be a finitely presented $A[t_1, \ldots, _n]$-module. Clearly, $R \cdot Q(M) \subseteq Q(M)$, so it suffices to show

$$\forall f_0, f_1 \in Q(M) : f := f_0 + f_1 \in Q(M).$$

Let $N := M/t_nM$ (which is f.p. over $A[t_1, \ldots, t_{n-1}]$) and $L := M/(t_1, \ldots, t_n)M$.

Apply (A_1) to $A[t_1, \ldots, t_{n-1}] \to A[t_1, \ldots, t_{n-1}][t_n]$ and M_f, so M_f is extended from N_f.

Claim. N_f is extended from L along $A_f \to A_f[t_1, \ldots, t_{n-1}]$.

Thanks to (B_{n-1}), it suffices to check that $(N_f)_\mathfrak{m}$ is extended from $(A_f)_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R_f)$.

Write $\mathfrak{p} := \mathfrak{m} \cap R$, i.e. $\mathfrak{m} = \mathfrak{p}_f$. Since $f \notin \mathfrak{p}$, we have $f_0 \notin \mathfrak{p}$ or $f_1 \notin \mathfrak{p}$, wlog. $f_0 \notin \mathfrak{p}$. But M_{f_0} **is** extended from L_{f_0}, so $(N_f)_\mathfrak{m} = N_\mathfrak{p}$ is extended from $L_\mathfrak{p}$.

This shows M_f is extended from A_f, so $f \in Q(M)$. Subsequently, $(A_1) \implies (A_n)$ for all n.

Step 3: Now we prove (A_1), i.e. for M a finitely presented $A[t]$-module, we show $f_0, f_1 \in Q(M) \implies f := f_0 + f_1 \in Q(M)$.

First we replace R by R_f, so we may assume $(f_0, f_1) = (1)$. With $N := M/tM$ we want to show $M \simeq N[t]$.

Let $u_i : M_{f_i} \xrightarrow{\sim} N_{f_i}[t]$ be isomorphisms for $i \in \{0, 1\}$. WLOG $u_i = \mathrm{id} \mod (t)$ (if not, postcompose with an automorphism of N_{f_i}). We have the following diagram:

$$
\begin{array}{ccccccc}
M_{f_0} & \xrightarrow{loc} & & M_{f_0 f_1} & & \xleftarrow{loc} & M_{f_1} \\
u_0 \downarrow \wr & & \swarrow^{(u_0)_{f_1}} \sim & & \sim \searrow^{(u_1)_{f_0}} & & \wr \downarrow u_1 \\
N_{f_0}[t] & \xrightarrow{loc} & N_{f_0 f_1}[t] & \underset{\sim}{\overset{\theta}{\dashrightarrow}} & N_{f_0 f_1}[t] & \xleftarrow{loc} & N_{f_1}[t]
\end{array}
$$

If the two isos $(u_0)_{f_1}$ and $(u_1)_{f_0}$ coincide, we can glue u_0 and u_1 together to an $A[t]$-isomorphism $M \xrightarrow{\sim} N[t]$. Therefore, we try to adjust the u_i to make this happen.

Lemma (Quillen's Elementary Fact about Localization)**.** *Let E be an R-algebra and $f_0, f_1 \in R$ such that $(f_0, f_1) = (1) = R$. Write $(1 + tE[t])^t imes := \{\alpha \in E[t]^\times \mid \alpha \equiv 1 \mod (t)\}$. Then*

$$(1 + tE_{f_0,f_1}[t])^\times = \big((1 + tE_{f_1}[t])^\times\big)_{f_0} \cdot \big((1 + tE_{f_0}[t])^\times\big)_{f_1}.$$

We apply this to $E := End_A(N)$.

$$\text{Let } \theta := (u_1)_{f_0} \circ {(u_0)_{f_1}}^{-1} \in \mathrm{End}_{A_{f_0 f_1}[t]}\left(N_{f_0 f_1}[t]\right) \simeq E_{f_0 f_1}[t].$$

In fact, $\theta = \mathrm{id} \mod (t)$, so $\theta \in (1 + tE_{f_0 f_1}[t])^\times$.

By the elementary localization lemma, $\theta = \theta_0 \cdot \theta_1$ with $\theta_0 \in \left((1 + tE_{f_1}[t])^\times\right)_{f_0}$ and $\theta_1 \in \left((1 + tE_{f_0}[t])^\times\right)_{f_1}$. Thus we find $v_i \in (1 + tE_{f_i}[t])^\times \subseteq Aut_{A_{f_i}[t]}\left(N_{f_i}[t]\right)$ with $\theta_0 = (v_1)_{f_0}^{-1}$ and $\theta_1 = (v_0)_{f_1}$. Then $(v_0 u_0)_{f_1} = \theta_1 (u_0)_{f_1}$ and $(v_1 u_1)_{f_0} = \theta_0^{-1}(u_1)_{f_0}$ and $(v_1 u_1)_{f_0} \circ {(v_0 u_0)_{f_1}}^{-1} = \theta_0^{-1}\theta_0\theta_1\theta_1^{-1} = \mathrm{id}$.

So we're done by replacing u_i with $v_i u_i$. $\square$

Proofs for the black boxes from commutative algebra

This is a sketch of the Nashier-Nichols proof of the Local Horrocks Theorem.

Let $(R, \mathfrak{m})$ be a local ring, $k := R/\mathfrak{m}$. For any R-module M we write $\overline{M} := M/\mathfrak{m}R$.

Lemma. *If an ideal $I \trianglelefteq R[t]$ contains a monic polynomial, then any monic $\gamma \in \overline{I}$ can be lifted to a monic in I.*

Proposition. *Let $I \trianglelefteq R[t]$ be an invertible ideal. If it contains a monic f, then $I = (g)$ for some monic $g \in I$. In particular, $R[t] \to I,\ 1 \mapsto g$ is an R-isomorphism.*

Lemma (Top-Bottom Lemma)**.** *Let $f = \sum a_i t^i, g = \sum b_j t^j \in R[t]$ such that $a_n, b_0 \in R^\times$. If $\{b_1, \ldots, b_n\} \subset \mathfrak{m}$, then $(f, g) = (1)$.*

Proof of Local Horrocks. First note that R has no nontrivial idempotents, hence $\operatorname{rk} P$ is constant. We do induction on $n := \operatorname{rk} P$. Denote by $S \subset R[t]$ the multiplicative set of monic polynomials.

Let $n = 1$ and $\varphi : P_S \to R[t]_S$ an $R[t]_S$-iso. Since P is finitely generated, we can modify φ such that $I := \varphi(P) \subseteq R[t]$. Since $I_S = R[t]_S$, the ideal I contains an element of S (a monic). Since $\varphi|_P : P \xrightarrow{\sim} I$, the module I is projective, hence an invertible ideal. We conclude $P \xrightarrow{\sim} I \xleftarrow{\sim} R[t]$.

For the inductive step let $n \geq 2$. Choose $p_1, \ldots, p_n \in P$ such that they form an $R[t]_S$-basis for P_S. Since $\overline{R[t]} = k[t]$ is a PID, $\overline{P}$ is $\overline{R[t]}$-free, and the theorem on elementary divisors over a PID tells us that there exist $\overline{q_1}, \ldots, \overline{q_n} \in \overline{P}$ that form a basis of $\overline{P}$ and satisfy $\overline{p_1} = \alpha \overline{q_2}$ for some $\alpha \in k[t]$.

Now set $p := q_1 + t^r p_1 \in P$, for $r \gg 0$, then $\overline{p} = \overline{q_1} + t^r \alpha \overline{q_2}$, so $\overline{p}, \overline{q_2}, \ldots, \overline{q_n}$ form a $\overline{R[t]}$-basis for $\overline{P}$.

Choose a monic $s \in S$ such that $sq_1 = \sum_{i=1}^n h_i p_i$ for some $h_i \in R[t]$. Then $sp = (h_1 + st^r)\, p_1 + \sum_{i=2}^n h_i p_i$. For $r \gg 0$, $(h_1 + st^r)$ is monic, so $p, p_2, \ldots, p_n$ form a $R[t]_S$-basis for P_S.

Now we study the multiplicative set $T := 1 + \mathfrak{m}R[t] \subset R[t]$.

Claim. $p, q_2, \ldots, q_n$ form a $R[t]_T$-basis for P_T.

Proof of Claim. From $1 + \mathfrak{m}R[t] \subset (R[t]_T)^\times$ follows $1 + \mathfrak{m}R[t]_T \subset (R[t]_T)^\times$, so $\mathfrak{m}R[t]_T \subset rad(R[t]_T)$.

$$\frac{P_T}{\mathfrak{m}R[t]_T P_T} \simeq \left(\frac{P}{\mathfrak{m}R[t]P}\right)_T = \left(\overline{P}\right)_T = \overline{P}.$$

Now $\overline{p}, \overline{q_2}, \ldots, \overline{q_n}$ is a $\overline{R[t]_T}$-basis for $\overline{P_T}$ and the Nakayama lemma (whose assumptions we just checked) tells us that $p, q_2, \ldots, q_n$ has to be a $R[t]_T$-basis for P_T.

Finally we take the $R[t]$-module $Q := P/pR[t]$. The localizations $Q_S \simeq P_S/pR[t]_S$ and $Q_T \simeq P_T/pR[t]_T$ are both free of rank $n-1$. The Top-Bottom-Lemma says that a maximal ideal of $R[t]$ avoids at least S or T, so we conclude that Q is locally free of rank $n-1$, hence Q is finitely generated projective of rank $n-1$.

From the induction hypothesis, $Q_S \simeq (R[t]_S)^{n-1} \implies Q \simeq (R[t])^{n-1}$, so

$$P \simeq R[t]p \oplus Q \simeq (R[t])^n\,. \qquad \square$$

Quillen actually proved the following localization lemma:

Theorem. Let E be a (not necessarily commutative) ring with 1, x, y, t commuting indeterminates over E and $E[x,y,t]^\times$ the group of invertible elements of $E[x,y,t]$ that have constant term 1. For a central element $f \in E$ we write $\left(E[x,y,t]^\times\right)_f$ for the image of $E[x,y,t]^\times \to E_f[x,y,t]^\times$.

For all $\theta(t) \in E_f[t]^\times$ there exists $k \geq 0$ such that

$$\theta\left(\left(x + f^k y\right) t\right) \theta(xt)^{-1} \in \left(E[x,y,t]^\times\right)_f.$$

Proof. Define $\varphi(x,y) \in E_f[x,y]$ by $\theta(x+y) - \theta(x) = y\varphi(x,y)$. For $r \geq 0$,

$$\begin{aligned}\theta\left(\left(x + f^r y\right) t\right) \theta(xt)^{-1} &= 1 + \left(\theta\left(\left(x + f^r y\right) t\right) \theta(xt)\right) \theta(xt)^{-1} \\ &= 1 + f^r yt\varphi(xt, f^r yt)\theta(xt)^{-1}.\end{aligned}$$

For $r \gg 0$, we have $f^r \varphi(x,y)\theta(x)^{-1} \in E[x,y]$. Consequently, $1 + f^r yt\varphi(xt, f^r yt)\theta(xt)^{-1} = \sigma(x,y,t)_f$ for some $\sigma(x,y,t) \in E[x,y,t]$. We can choose σ such that $\sigma(x,y,t) = 1 \mod (yt)$. If $\sigma(x,y,t)$ would be invertible, we would be done.

In $E_f[x,y,t]$, the inverse of $\sigma(x,y,t)$ is

$$\theta(xt)\theta\left(\left(x + f^r y\right) t\right)^{-1} = \sigma(x + f^r y, -y, t)_f.$$

Define $\sigma'(x,y,t) := \sigma(x + f^r y, -y, t) \in E[x,y,t]$, then we have $\sigma^{-1} = \sigma'$ in $E_f[x,y,t]$. Since $\sigma'(x,y,t) = 1 \mod (yt)$ we can write

$$\begin{aligned}\sigma\sigma' &= 1 + yt\mu_1, \quad \text{for } \mu_1 \in E[x,y,t], \\ \sigma'\sigma &= 1 + yt\mu_2, \quad \text{for } \mu_2 \in E[x,y,t].\end{aligned}$$

Since $\sigma\sigma' = 1 = \sigma'\sigma$ after localization at f, we find $s \gg 0$ such that $f^s\mu_1 = 0 = f^s\mu_2$. Consequently, $\sigma(x, f^s y, t) \in E[x,y,t]^\times$ with inverse $\sigma'(x, f^s y, t)$. We replace r by $k := r+s$ and get

$$\theta\left(\left(x + f^r y\right) t\right) \theta(xt)^{-1} = \sigma(x, f^s y, t)_f \in \left(E[x,y,t]^\times\right)_f. \qquad \square$$

Corollary. Let E be an R-algebra and $f \in R$, and $\theta(t) \in E_f[t]^\times$. Then there exists $k \geq 0$ such that for any $a, b \in R$ with $a - b \in (f^k)$ we have $\theta(at)\theta(bt)^{-1} \in \left(E[t]^\times\right)_f$.

Proof. Pick $x := b$ and $y := (a-b)f^{-k}$ in the theorem. $\square$

Corollary. Let E be an R-algebra and $f_0, f_1 \in R$ such that $(f_0, f_1) = (1)$. Then

$$E_{f_0 f_1}[t]^\times = \left(E_{f_1}[t]^\times\right)_{f_0} \cdot \left(E_{f_0}[t]^\times\right)_{f_1}.$$

Proof. Apply the previous corollary to $\theta(t)$ and each of the localizations $E_{f_1} \to (E_{f_1})_{f_0}$ and $E_{f_0} \to (E_{f_0})_{f_1}$, then pick a $k \geq 0$ that works for both.

From $(f_0, f_1) = (1)$ follows $(f_0^k, f_1^k) = (1)$: if $(f_0^k, f_1^k) \subset \mathfrak{n} \subset (1)$ for $\mathfrak{n}$ a prime ideal, then $f_0^k \in \mathfrak{n} \implies f_0 \in \mathfrak{n}$, so $(f_0, f_1) \in \mathfrak{n}$, hence $(1) \subset \mathfrak{n}$. This shows that (f_0^k, f_1^k) is not contained in any proper prime ideal, hence $(f_0^k, f_1^k) = (1)$.

Now we can pick $b \in (f_1^k)$ such that $1 - b \in (f_0^k)$, then

$$\theta(t)\theta(bt)^{-1} \in \left(E_{f_1}[t]^\times\right)_{f_0} \text{ and } \theta(bt)\theta(0)^{-1} \in \left(E_{f_0}[t]^\times\right)_{f_1}. \qquad \square$$