The Smash Product of CW-Spectra

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2013-06-24

You could have invented smash products!

- What do we expect, what do we want?
- Naive smash products of spectra:
  $E \wedge_{ab} F$ for series a_n, b_n of natural numbers.
- Some properties of some naive smash products.

The less naive smash product.

- $E \wedge F$ is defined via a 2-dimensional telescope over all naive smash products $E \wedge_{ab} F$.

- List of all properties one usually proves about this smash product (that certain diagrams commute up to homotopy).

- Construction of a map $eq_{ab} : E \wedge_{ab} F \hookrightarrow E \wedge F$ which is reasonably often a homotopy equivalence, by embedding a 1-dimensional telescope over $E \wedge_{ab} F$ into $E \wedge F$.

- Use various naive $E \wedge_{ab} F$ and the eq_{ab} to prove the properties of $E \wedge F$.

What do we expect?

Generalizing the smash product of spaces

We want the smash product of two spectra to generalize the smash product of a spectrum with a space. One way to say this: $E \wedge S^\infty X$ should be homotopy equivalent to $E \wedge X$. In particular, $S^\infty X \wedge S^\infty Y$ should be homotopy equivalent to $S^\infty(X \wedge Y)$. Another special case is that we have homotopy equivalences $l : S^0 \wedge E \xrightarrow{\sim} E$ and $r : E \wedge S^0 \xrightarrow{\sim} E$, which should be natural in E if we take the smash product in the sense of spectra, as well.

What do we expect?

Suspension

Smashing a spectrum with the suspension of a space means smashing with S^1 and then with the space. Therefore, smashing a spectrum with the suspension of a spectrum should be no more different. On spectra, we can also formally desuspend (i.e. take Σ^{-1}). We expect

$$\forall k \in \mathbb{Z} : (\Sigma^k E) \wedge F \simeq \Sigma^k (E \wedge F).$$

What do we expect?

Natural transposition maps

For spaces, X, Y, the smash product admits a transposition map $\tau : X \wedge Y \to Y \wedge X$, which is often non-trivial. Example: $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ is homotopy equivalent to $\nu \wedge \mathrm{id}$, with ν the inversion on S^1. We expect such a map and expect it to be natural and non-trivial for spectra as well.

What do we want?

Associativity up to homotopy

The smash product of spaces is associative, i.e. $X \wedge (Y \wedge Z)$ is homeomorphic to $(X \wedge Y) \wedge Z$. This is quite hard to achieve for spectra, so we relax our expectations and require a "good" smash product of spectra to be associative only up to homotopy, i.e. there is a homotopy equivalence $a : E \wedge (F \wedge G) \to (E \wedge F) \wedge G$.

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Different ways to associate

There are various ways of re-ordering brackets:
$((ef)g)h = (ef)(gh) = e(f(gh))$ or
$((ef)g)h = (e(fg))h = e((fg)h) = e(f(gh))$. If we just have a homotopy equivalence a instead of "=", those two ways might be non-identical, so we can not just omit brackets.

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If we require the two ways of re-ordering brackets to be homotopy equivalent again, then we can omit all brackets (at least in the homotopy category).

The naive smash product(s)
Very naive

An obvious generalization

Let E be a spectrum, X a space and $S^\infty X$ its suspension spectrum. Then we could define $E \wedge S^\infty X$ by

$$(E \wedge S^\infty X)_n := E_n \wedge (S^\infty X)_0 = E_n \wedge X$$

and the obvious structure maps, to recover $E \wedge S^\infty X = E \wedge X$.

Problem

If we suspend $S^\infty X$ once, we get $(\Sigma S^\infty X)_0 = *$, so the very naive smash product just defined gives $*$ as well, which is not the suspension of $E \wedge X$.

The naive smash product(s)
Still naive

Something you could have invented

Let E, F be two spectra. Define

$$(E \wedge F)_{2n} := (E_n \wedge F_n) \text{ and } (E \wedge F)_{2n+1} := (E_{n+1} \wedge F_n).$$

Denote the structure maps of E and F by $e : \Sigma E_n \to E_{n+1}$ and $f : \Sigma F_n \to F_{n+1}$, then we define $\Sigma(E \wedge F)_n \to (E \wedge F)_{n+1}$ by

$$S^1 \wedge E_n \wedge F_n \xrightarrow{e \wedge 1} E_{n+1} \wedge F_n$$

for even n; for odd n we define it by

$$S^1 \wedge E_n \wedge F_n \xrightarrow{\tau \wedge 1} E_n \wedge S^1 \wedge F_n \xrightarrow{1 \wedge f} E_n \wedge F_{n+1}.$$

This defines a spectrum!

The naive smash product(s)
A whole series of constructions

The general naive smash product

Let $a_n, b_n : \mathbb{N} \to \mathbb{N}$ be two monotone increasing sequences ($a_{n+1} \geq a_n$) of nonnegative integers with the extra property

$$\forall n : a_n + b_n = n.$$

Using this, we define a smash product of two spectra E and F

$$(E \wedge_{ab} F)_n := E_{a_n} \wedge F_{b_n}$$

with structure maps similar to the previous example.

Examples

With $a_n = \lceil n/2 \rceil$ and $b_n = \lfloor n/2 \rfloor$ we get back the previous special case. With $a_n = n$ and $b_n = 0$ we get back the very naive smash product.

The naive smash product(s)
If you look in the book...

Switzer's notation
Switzer denotes $E \wedge_{ab} F$ by $E \wedge_{AB} F$, where $A \sqcup B = \mathbb{N}$ is a partition of the nonnegative integers.

To get a_n, b_n from these, Switzer defines $a_n := |\{a \in A \mid a < n\}|$.

To get $A \sqcup B$ from a_n, b_n, we define $A := \{n \in \mathbb{N} \mid a_{n+1} \neq a_n\}$.

There are reasons for using Switzer's notation, in particular if we want to do something like $A \sqcup B = C$ with C order isomorphic to $\mathbb{N}$ and then $C \sqcup D = \mathbb{N}$. This is useful for considering triple smash products, but not necessary.

Functions

Given functions of spectra $\phi : E \to G$, $\psi : F \to H$, we can form

$$\phi \wedge_{ab} \psi : E \wedge_{ab} F \to G \wedge_{ab} H$$

by defining it on $(E \wedge_{ab} F)_n = E_{a_n} \wedge F_{b_n}$ to be $\phi_{a_n} \wedge \psi_{b_n}$. This commutes with the structure maps for $E \wedge_{ab} F$, since these are either $e \wedge 1$ or $1 \wedge f$.

Functoriality

Morphisms

Given morphisms of spectra $\Phi : E \to G$, $\Psi : F \to H$ represented by functions on cofinal subspectra $\phi : E' \to G$, $\psi : F' \to H$, we can certainly form $\phi \wedge_{ab} \psi : E' \wedge_{ab} F' \to G \wedge_{ab} H$.

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Problem

In general $E' \wedge_{ab} F'$ is not cofinal in $E \wedge_{ab} F$.

Solution

Consider only sequences $a_n \xrightarrow{n \to \infty} \infty$, then $E' \wedge_{ab} F$ is cofinal in $E \wedge_{ab} F$.

Functoriality

On the homotopy category

For E, F spectra, X a space and $a_n, b_n \to \infty$, we have

$$E \wedge_{ab} (F \wedge X) \simeq (E \wedge_{ab} F) \wedge X \simeq (E \wedge X) \wedge_{ab} F.$$

In particular, this holds for $X = I^+$, so the homotopy class of a morphism $\Phi \wedge_{ab} \Psi$ depends only on the homotopy classes of Φ and Ψ.

Therefore, $E \wedge_{ab} F$ is also a functor on the homotopy category of spectra.

Properties of certain naive smash products

Left and right units

To get homotopy equivalences $l : S^0 \wedge_{ab} E \to E$, we can consider $a_n := 0$ and $b_n := n$ for the left unit, then $(S^0 \wedge_{ab} E)_n = S^0 \wedge E_n$ and we already have a homotopy equivalence $l : S^0 \wedge X \to X$, $(s, x) \mapsto x$ for any space X. These homotopy equivalences $l_n : S^0 \wedge E_n \to E_n$ commute with the structure map, since the structure map on $S^0 \wedge_{ab} E$ was defined as $1 \wedge e$.

To get homotopy equivalences $r : E \wedge_{ab} S^0 \to E$, we do the same with a and b in reversed roles.

Properties of certain naive smash products

Associativity

Let a_n, b_n, c_n be diverging sequences of nonnegative integers with $a_n + b_n + c_n = n$.

Choose subsequences a', b' of a, b such that $a'_{a_n+b_n} = a_n$ and $b'_{a_n+b_n} = b_n$. (Let k'_n be a monotone increasing sequence such that $a_{k'_n} + b_{k'_n} = n$ and denote $a'_n := a_{k'_n}$, $b'_n := b_{k'_n}$. Notice that $a'_{a_n+b_n} + b'_{a_n+b_n} = a_n + b_n$).

Similarly, we get for b and c the sequences b'' and c''.

With this notational setup, we have

$$((E \wedge_{a'b'} F) \wedge_{a+b,c} G)_n = (E \wedge_{a'b'} F)_{a_n+b_n} \wedge G_{c_n} = (E_{a_n} \wedge F_{b_n}) \wedge G_{c_n}$$

$$(E \wedge_{a,b+c} (F \wedge_{b''c''} G))_n = E_{a_n} \wedge (F \wedge_{b''c''} G)_{b_n+c_n} = E_{a_n} \wedge (F_{b_n} \wedge G_{c_n})$$

and there is a homeomorphism from the first to the second triple smash product of spaces, which we call a.

Since a is natural, this gives a morphism of spectra.

Properties of certain naive smash products

Associativity and Units

One would expect the following diagram to commute:

$$(S^0 \wedge E) \wedge F \xrightarrow{\quad a \quad} S^0 \wedge (E \wedge F)$$

with maps $l \wedge 1$ and l going down to $E \wedge F$.

but in our construction of l we have used sequences which don't go to infinity, while our construction of a required sequences which do go to infinity.

This is a problem the naive smash product won't solve!

Properties of certain naive smash products

Cofibre sequences

Given a cofibre sequence $E \to F \to G$ of spectra, and a spectrum H, we can form $E \wedge_{ab} H \to F \wedge_{ab} H \to G \wedge_{ab} H$ and this is again a cofibre sequence.

We can easily prove this for a special cofibre sequence $E \xrightarrow{\Phi} F \to F \cup_{\Phi} CE$ by showing that

$$(F \cup_{\Phi} CE) \wedge_{ab} H = (F \wedge_{ab} H) \cup_{(\Phi \wedge 1)} C(E \wedge_{ab} H),$$

which is clear in each degree n, as long as we use sequences $a, b \to \infty$.

The less naive smash product.

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- Construction of a map $eq_{ab} : E \wedge_{ab} F \hookrightarrow E \wedge F$ which is reasonably often a homotopy equivalence, by embedding a 1-dimensional telescope over $E \wedge_{ab} F$ into $E \wedge F$.

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The less naive smash product

First, a reminder on telescopes

We shall later on need the 1-dimensional telescope over a spectrum E. For this, we have a "base" space $\mathbb{R}_{\geq 0}$ which we think of as union of intervals $[i, i+1]$ for $i \in \mathbb{Z}_{\geq 0}$. For the construction of $T(E)_n$ we use only the union of all intervals with left corner $i \leq n$.

To construct $T(E)_n$, over an interval $[i, i+1]$ we take $S^{n-i} \wedge E_i$ and identify over a point $\{i+1\}$ with $i \in \mathbb{Z}_{\geq 0}$ the space $S^{n-i} \wedge E_i = S^{n-i-1} \wedge S^1 \wedge E_i$ with the subspace of $S^{n-(i+1)} \wedge E_{i+1}$ via $1 \wedge e$, where $e : S^1 \wedge E_i \to E_{i+1}$ is the structure map of E. Formally, this is

$$T(E)_n := \left(\left(\bigvee_{i=0}^{n} S^{n-i} \wedge E_i \wedge \{i\}^+ \right) \vee \left(\bigvee_{i=0}^{n-1} S^{n-i} \wedge E_i \wedge [i, i+1]^+ \right) \right) / \sim$$
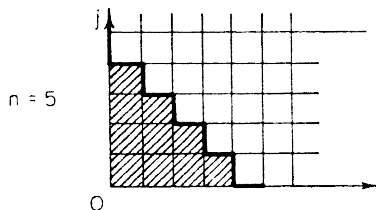
The less naive smash product

The base space for a 2-dimensional telescope

We give the half-open square $Q := \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ the structure of a CW complex, with 0-cells the points $(i, j) \in Q$ with $i, j \in \mathbb{Z}_{\geq 0}$, the 1-cells the intervals $[i, i+1] \times \{j\} \subset Q$ and the intervals $\{i\} \times [j, j+1]$ with $i, j \in \mathbb{Z}_{\geq 0}$, and 2-cells the closed squares $[i, i+1] \times [j, j+1] \subset Q$ with $i, j \in \mathbb{Z}_{\geq 0}$.

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A filtration on Q by subcomplexes

Let Q_n be the subcomplex of Q which consists only of the cells e with lower left corner (i, j) such that $i + j - dim(e) \leq n$. Then Q_n looks like a stair:

The less naive smash product

Construction of $(E \wedge F)_n$ over the 0- and 1-cells

For $(E \wedge F)_n$ we take over the 0-cells of Q_n the spaces
$S^{n-i-j} \wedge E_i \wedge F_i \wedge \{(i,j)\}^+$.
Over the 1-cells of Q_n we take

$$S^{n-i-j} \wedge E_i \wedge F_i \wedge (\{i\} \times [j, j+1])^+ \text{ and}$$

$$S^{n-i-j} \wedge E_i \wedge F_i \wedge ([i, i+1] \times \{j\})^+,$$

where we have to make the obvious identifications of the part over the 0-cells with the "edge" over the part over the 1-cells, via the structure maps of the spectra E and F.

What we need to do

We have to define something over a 2-cell $e = [i, i+1] \times [j, j+1]$ which is consistent with our previous definition on the boundary ∂e. Observe that ∂e consists of two paths from (i, j) to $(i+1, j+1)$, and our identifications made for 1-cells correspond to two maps (going first up then right and going first right then up)

$$S^{n-i-j} \wedge E_i \wedge F_j \to S^{n-(i+1)-(j+1)} \wedge E_{i+1} \wedge F_{j+1},$$

which don't coincide.

If we want to "fill in" something over e, it should have the same monodromy as our construction over ∂e.

The monodromy of the construction over 1-cells

Let $(s, t_1, t_2, x, y) \in S^{n-i-j-2} \wedge S^1 \wedge S^1 \wedge E_i \wedge F_j$, then we get via going first up then right:

$$\mapsto (s, t_1, x, t_2, y) \mapsto (s, t_1, x, f(t_2, y)) \mapsto (s, e(t_1, x), f(t_2, y))$$

and from going first right then up:

$$\mapsto (s, t_1, e(t_2, x), y) \mapsto (s, e(t_2, x), t_1, y) \mapsto (s, e(t_2, x), f(t_1, y)),$$

so the difference between the two is precisely a precomposition with an appropriate switch map $1 \wedge \tau \wedge 1 \wedge 1$.

The less naive smash product
Construction of $(E \wedge F)_n$ over the 2-cells

A bundle-theoretic description

Let ξ be the vector bundle on $S^1 \wedge S^1$ with transition map the switch map $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$. Then our construction of $(E \wedge F)_n$ over ∂e is isomorphic to $S^{n-i-j-2} \wedge M(\xi) \wedge E_i \wedge F_j$, with $M(\xi)$ the Thom space of ξ.

Extending to the 2-cells

Because of $\pi_1(BSO(2)) = \pi_0(SO(2)) = *$, we can deform the classifying map of ξ to a constant map $S^1 \to BSO(2)$, which obviously extends to D^1, thus gives rise to a bundle on e which extends ξ.

We can thus define $S^{n-i-j-2} \wedge M(\xi) \wedge E_i \wedge F_j$ over the 2-cell e, and then make obvious identifications with the 1-cells and 0-cells we already have.

The less naive smash product

Structure maps

The maps $\Sigma(E \wedge F)_n \to (E \wedge F)_{n+1}$ are given for each 0-cell (i,j) by the identity

$$S^1 \wedge (S^{n-i-j} \wedge E_i \wedge F_j) \to S^{n+1-i-j} \wedge E_i \wedge F_j.$$

We have already built in the structure maps of E and F in our identifications that make up $(E \wedge F)_n$.

Properties

Theorem
$E \wedge F$ is functorial in E and F and there are homotopy equivalences

$$a = \qquad a_{E,F,G} : (E \wedge F) \wedge G \to E \wedge (F \wedge G)$$
$$\tau = \qquad \tau_{E,F} : E \wedge F \to F \wedge E$$
$$l = \qquad l_E : S^0 \wedge E \to E$$
$$r = \qquad r_E : E \wedge S^0 \to E$$
$$\sigma = \qquad \sigma_{E,F} : (\Sigma E) \wedge F \to \Sigma(E \wedge F)$$

which are natural in the homotopy category, such that the following 8 diagrams commute up to homotopy.

Diagrams I

i) (associativity)

$$((E \wedge F) \wedge G) \wedge H \xrightarrow{a} (E \wedge F) \wedge (G \wedge H) \xrightarrow{a} E \wedge (F \wedge (G \wedge H))$$

$$((E \wedge F) \wedge G) \wedge H \xrightarrow{a \wedge 1} (E \wedge (F \wedge G)) \wedge H \xrightarrow{a} E \wedge ((F \wedge G) \wedge H) \xrightarrow{1 \wedge a}$$

ii) (transposition)

Diagrams II

iii) (associativity and transposition)
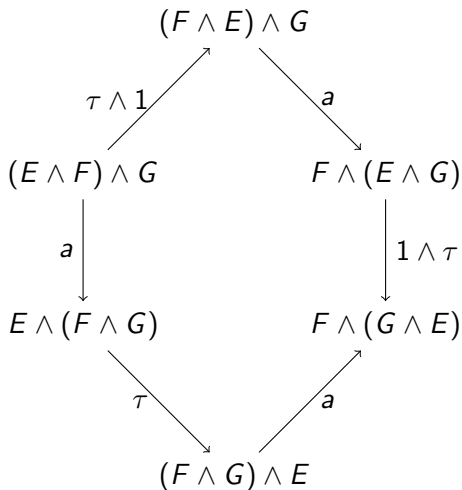
Diagrams III

iv) (associativity and units)

$$(S^0 \wedge E) \wedge F \xrightarrow{\quad a \quad} S^0 \wedge (E \wedge F)$$

$l \wedge 1$

l

$$E \wedge F$$

v) (associativity and units)

$$(E \wedge S^0) \wedge F \xrightarrow{\quad a \quad} E \wedge (S^0 \wedge F)$$

$r \wedge 1$

$1 \wedge l$

$$E \wedge F$$

Diagrams IV

vi) (associativity and units)

$$(E \wedge F) \wedge S^0 \xrightarrow{\quad a \quad} E \wedge (F \wedge S^0)$$

r

$1 \wedge r$

$$E \wedge F$$

vii) (associativity and units)

$$S^0 \wedge E \xrightarrow{\quad a \quad} E \wedge S^0$$

l

r

$$E$$
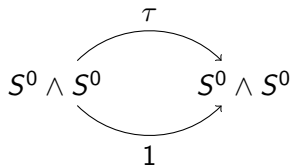
Diagrams V

viii) (transposition)

Why these eight diagrams?

Theorem (Mac Lane)

If you write down a smash product of some spectra (with parentheses!) and two ways of re-grouping brackets, transposing, applying units and suspension that reach the same conclusion, and these two ways are homotopy equivalent –
then there is a proof of this homotopy equivalence that uses just these 8 diagrams we've just seen.

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these two ways are homotopy equivalent –*
*then there is a proof of this homotopy equivalence that uses just
these 8 diagrams we've just seen.*

Proof.

Basically, you connect each expression in spectra to a canonical form
and each "path" between two expressions to a canonical path. These
are formed by the 8 diagrams. □

Properties

Furthermore

- If X is a space with suspension spectrum $S^\infty X$ and E any spectrum, then $E \wedge S^\infty X \simeq E \wedge X$.
- If $E \to F \to G$ is a cofibre sequence and H any spectrum, then $E \wedge H \to F \wedge H \to G \wedge H$ is a cofibre sequence.
- For spectra E_i, $i \in I$, we have a natural homotopy equivalence $(\bigvee_i E_i) \wedge F \to \bigvee_i (E_i \wedge F)$.

Relating the naive and the less naive

Proposition
There is a homotopy equivalence $\rho : T(E \wedge_{ab} F) \to E \wedge_{ab} F$.

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Proof.

Remember, for a spectrum E we defined

$$T(E)_n = \left(\left(\bigvee_{i=0}^{n} S^{n-i} \wedge E_i \wedge \{i\}^+ \right) \vee \left(\bigvee_{i=0}^{n-1} S^{n-i} \wedge E_i \wedge [i, i+1]^+ \right) \right) / \sim$$

so we can map $T(E)_n \to E_n$ by mapping each wedge factor $S^{n-i} \wedge E_i \to E_i + n - i$ via the $(n-i)$-fold structure map. This is obviously compatible with the structure maps of $T(E)_n$ and E_n. This is a homotopy equivalence, since it is a weak homotopy equivalence of CW spectra. Remark: the homotopy inverse is not easy to describe.

Now apply this to the spectrum $E \wedge_{ab} F$. $\qquad \square$

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Proof.
You can think of a_n, b_n as describing a path $\omega : \mathbb{R}_{\geq 0} \to Q$ that is piecewise linear:

$$\text{for } t \in [n, n+1] : \ \omega(t) := \begin{cases} (a_n + t - n, b_n) & \text{if } (a_{n+1} \neq a_n) \\ (a_n, b_n + t - n) & \text{if } (b_{n+1} \neq b_n) \end{cases}$$

This path lands in (a filtered subcomplex of) the 1-skeleton of Q. We can therefore map $(s, e, f, t) \in S^{n-k} \wedge E_{a_k} \wedge F_{b_k} \wedge [k, k+1]^+$ to $(s, e, f, \omega(t)) \in S^{n-a_k-b_k} \wedge E_{a_k} \wedge F_{b_k} \wedge \omega([k, k+1])^+$. $\qquad\square$

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Lemma

The map eq_{ab} is a homotopy equivalence if any of the following is satisfied:

1. $a_n, b_n \to \infty$,
2. $a_n \to d$ and $\forall r \geq d : \Sigma E_r = E_{r+1}$ for some $d \in \mathbb{N}$,

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The map eq_{ab} is a homotopy equivalence if any of the following is satisfied:

1. $a_n, b_n \to \infty$,
2. $a_n \to d$ and $\forall r \geq d : \Sigma E_r = E_{r+1}$ for some $d \in \mathbb{N}$,
3. $b_n \to d$ and $\forall r \geq d : \Sigma F_r = F_{r+1}$ for some $d \in \mathbb{N}$.

Relating the naive and the less naive

(Lemma: *eq* is a homotopy equivalence)

Proof.

First, we prove the case of assumption 1:

- For $n \in \mathbb{N}$, let $G_n \subset (E \wedge F)_n$ be the subcomplex over cells e_{ij} with lower left corner (i, j) such that $i \leq a_n$ and $j \leq b_n$. This gives a subspectrum $G \subset E \wedge F$. Assumption 1) shows G is cofinal.

- The inclusion $T(E \wedge_{ab} F)_n \to (E \wedge F)_n$ has image in G_n.

- There is a deformation retraction $G_n \twoheadrightarrow E_{a_n} \wedge F_{b_n}$ which is compatible with the structure maps.

Relating the naive and the less naive
(Lemma: *eq* is a homotopy equivalence)

Proof (cont.)

- In the diagram

$$
\begin{array}{ccc}
 & & T(E \wedge_{ab} F)_n \\
 & \overset{\rho}{\underset{\sim}{\longleftarrow}} & \big\downarrow \\
E_{a_n} \wedge F_{b_n} & & \\
 & \overset{\sim}{\longleftarrow} & G_n
\end{array}
$$

  the horizontal morphism must be a weak homotopy equivalence, too.

- This shows that $T(E \wedge_{ab} F) \hookrightarrow G \hookrightarrow E \wedge F$ is a homotopy equivalence.

Proof (cont.)

Now the case of assumption 2:

- We use a different subcomplex $G \subset E \wedge F$, in which we include all cells e_{ij} with lower left corner (i, j) such that $j \leq b_n$ and $i + j \leq n$. This allows for (i, j) with $i > a_n$, which is necessary to get a cofinal subcomplex.
- The rest of the argument is the same.
- The proof for assumption 3 is also the same.

□

Proving the main theorem

General strategy

For each diagram (i–viii), pick certain sequences a_n, b_n, c_n such that the naive smash products satisfy the diagrams and use the previous lemma to get the property for the not-naive smash product.

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For the diagrams (iv,v,vi) we need to be able to handle sequences with $a_n \to d$, so we can not construct the associator map a by using only sequences $a_n \to \infty$.

For (vii) we need we need to handle sequences with $a_n \to d$, so τ has to be constructed in a way that allows this.

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Solution

Employ more telescopes!

Thank you for your attention!